Applic. max pop? over time

intro one sided limits from 1.6 here!

1.5 ①

Section 1.5 Limits (2 classes to cover)

Limits are a concept useful in the development of Calculus (Diff. from Algebra)

Intuitive

What happens to $y=f(x)$ as x approaches zero.

ex) $f(x)=\frac{x^3+1}{x+1}$ Let's look at a table

x	$-.01$	$-.001$	0	$.001$	$.01$
$f(x)$			1		

$\lim_{x\to c^-} f(x)=L$

$\lim_{x\to c^+} f(x)=L$

If $\lim = L$ then $\lim_{x\to c} f(x)$ exists & $=L$

As x gets closer to 0, $f(x)$ gets closer to 1.

Limit Notation $\lim_{x\to 0} f(x)=1$

If $f(x)$ gets closer & closer to L as x gets closer & closer to c from both sides, then L is the limit of $f(x)$ as x approaches c.

$\lim_{x\to c} f(x)=L$

ex) Given $f(x)=\frac{x^3+1}{x+1}$ find $\lim_{x\to -1} f(x)$ using a table

x	-1.01	-1.001	-1	-0.999	-0.99
$f(x)$	3.0301	3.003	X	2.997	2.9701

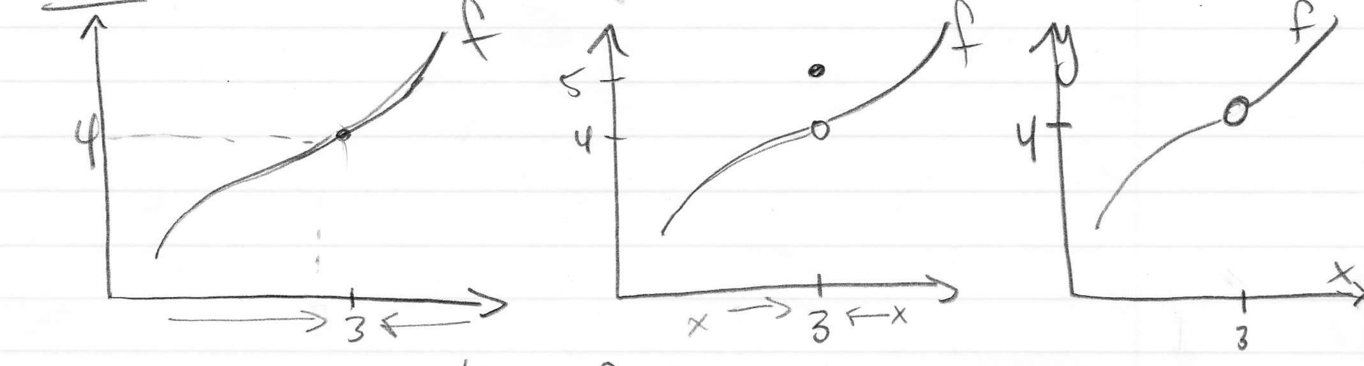
$$\boxed{\lim_{x\to -1} f(x)=3}$$

ex) $f(x)=x-\frac{1}{x}$ $\lim_{x\to 0} f(x)$

x	-0.09	-0.009	0	0.0009	0.009
$f(x)$	11.0211	111.102	X	-1111.11	-111.602

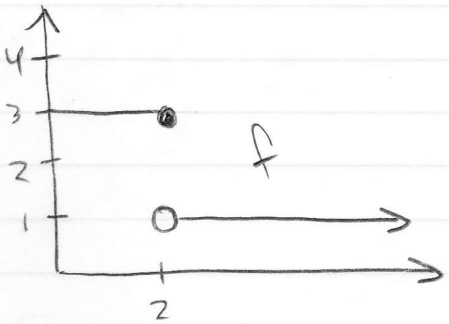
$\lim_{x\to 0} f(x) = \text{DNE}$

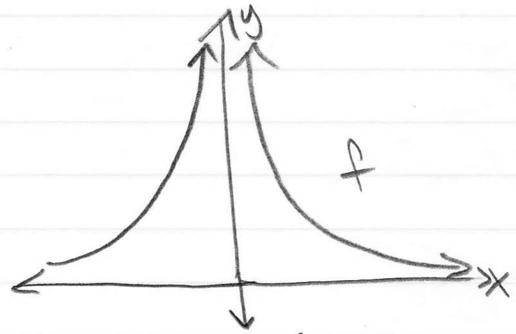
Limits near the point c, not at the point c.



$\lim_{x \to 3} f(x) = 4$

Limits that Do Not Exist (D.N.E.)

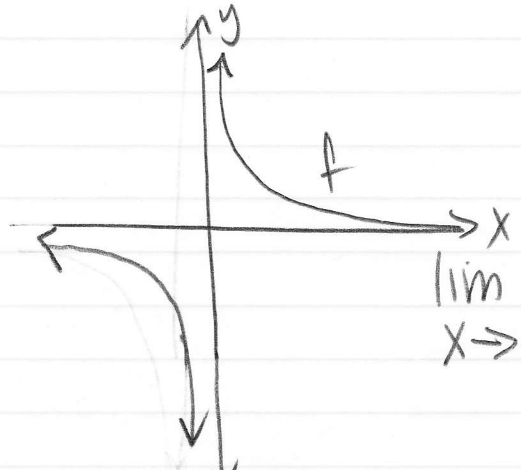




$\lim_{x \to 2} f(x) =$ DNE

$\lim_{x \to 0} f(x) = \infty$

The limit doesn't exist.

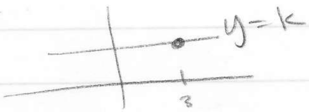


$\lim_{x \to 0} f(x) =$ D.N.E.

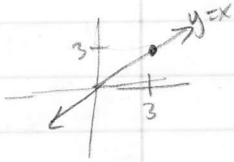
We can use tables / graphs or ... 1.5 (3)

Rules

(1) If k is a constant



- $\lim\limits_{x \to c} k = k$ ex. $\lim\limits_{x \to 2} 3 = 3$



- $\lim\limits_{x \to c} kf(x) = k \lim\limits_{x \to c} f(x)$ ex. not yet
- $\lim\limits_{x \to c} x = c$ ex. $\lim\limits_{x \to 4} x = 4$

(2) $\lim\limits_{x \to c} [f(x) \pm g(x)]$ (limit of sum is the sum of the limit)

(3) $\lim\limits_{x \to c} [f(x) \cdot g(x)] = \lim\limits_{x \to c} f(x) \lim\limits_{x \to c} g(x)$

ex) $\lim\limits_{x \to 2} x(x-1) = \lim\limits_{x \to 2} x \lim\limits_{x \to 2} x-1$ as $x \to 1$ $f(x) \to 0$ zero of $x(x-1)$!! :)

$= 2 \left(\lim\limits_{x \to 2} x - \lim\limits_{x \to 2} 1 \right)$

$= 2(2-1) = 2(1) = 2$

(4) $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)}$ ex. $\lim\limits_{x \to 5} \dfrac{4}{x} = \dfrac{\lim\limits_{x \to 5} 4}{\lim\limits_{x \to 5} x} = \boxed{\dfrac{4}{5}}$

(5) If $p(x)$ is a poly. just evaluate

$\lim\limits_{x \to c} p(x) = p(c)$

ex) $\lim\limits_{x \to -2} x^2 + 3x + 1 = (-2)^2 + (3)(-2) + 1$ once you plug value in, drop the limit.

$= 4 - 6 + 1$

$= -2 + 1 = \boxed{-1}$

(6) For any $k \in \mathbb{R}$

$\lim\limits_{x \to c} [f(x)]^k = \left[\lim\limits_{x \to c} f(x) \right]^k$ provide A exists

ex. $\lim\limits_{x \to 3} (x^2+1)^2 = (\qquad)^2$

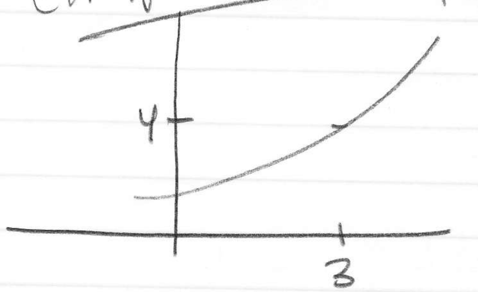
OPTIONAL

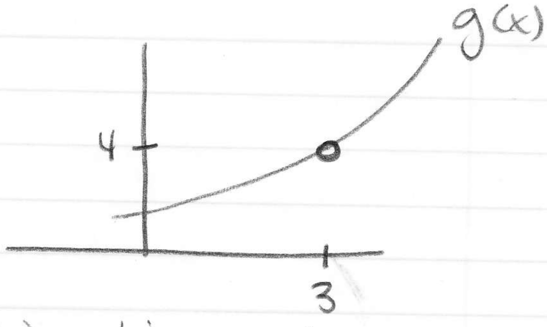
(7) $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} g(x)$ if $f(x) = g(x)$ $\forall x \neq c$

1.5 (3a)

OPTIONAL rule 7 justification

ex. Graphically





$$\lim_{x\to 3} f(x) = \lim_{x\to 3} g(x) = 4$$

ex. Algebraically

If $f(x) = \dfrac{x^2-4}{x-2}$ & we want to know

$$\lim_{x\to 2} \frac{x^2-4}{x-2}$$

*Always try to plug it in!

$$= \frac{(2)^2-4}{2-2} = \frac{0}{0} \leftarrow \text{No good}$$

But we can simplify f(x) 1st.

$$f(x) = \frac{x^2-4}{x-2} = \frac{(x+2)\cancel{(x-2)}}{\cancel{x-2}}$$

$$= x+2$$

This is like g(x) in rule 7. We can take the limit of this "new" function and it will equal the limit of original f(x).

$$\lim_{x\to 2} x+2 = 2+2$$

$$\boxed{=4}$$

We can do all of this inside the limit too.

11am

ex) $\displaystyle \lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x+2)\cancel{(x-2)}}{\cancel{x-2}}$

$$= \lim_{x\to 2} x+2$$

$$= 2+2 \boxed{=4}$$

Carry lim. around until you plug the value in.

1.5 (4)

Limits of polynomials & Rationals

(2) ex. Use properties of limits to evaluate.

$$\lim_{x \to -1} (3x^3 - 4x + 8)$$

Polynomial

$$= 3(-1)^3 - 4(-1) + 8$$

$$= 3(-1) + 4 + 8$$

$$= -3 + 12$$

$$\boxed{= 9}$$

Now lets verify, by example using the other limit rules.

Same

(3) Same problem

$$\lim_{x \to -1} (3x^3 - 4x + 8)$$

$$= \lim_{x \to -1} 3x^3 - \lim_{x \to -1} 4x + \lim_{x \to -1} 8 \quad = 8$$

$$= 3 \lim_{x \to -1} x^3 - 4 \lim_{x \to -1} x + 8$$

$$= 3 \left(\lim_{x \to -1} x\right)^3 - 4(-1) + 8$$

$$= 3(-1)^3 + 4 + 8$$

$$= 3(-1) + 4 + 8$$

$$= -3 + 4 + 8$$

$$= -3 + 12$$

$$\boxed{= 9}$$

Insert here

(1) ex. Given $\lim_{x \to 2} f(x) = 3$ & $\lim_{x \to 2} g(x) = 4$,

Calculate

$$\lim_{x \to 2} [f(x) + 5g(x)]$$

* Use limit rules

$$= \lim_{x \to 2} f(x) + \lim_{x \to 2} 5g(x)$$

$$= 3 + 5 \lim_{x \to 2} g(x)$$

$$= 3 + 5(4) = \boxed{23}$$

1pm Extra Example.

$$\lim_{x \to 3} \frac{x^2 - x - 1}{\sqrt{x+1}}$$

$$= \frac{\lim_{x \to 3} x^2 - x - 1}{\lim_{x \to 3} \sqrt{x+1}} = \frac{3^2 - 3 - 1}{\sqrt{\lim_{x \to 3} x+1}} = \frac{9 - 4}{\sqrt{3+1}} = \frac{5}{\sqrt{4}} = \boxed{\frac{5}{2}}$$

Rationals

Technique

1. Try to plug in $x=c$
2. Simplify by
   - Factoring
   - Get common denom & add.
   - mult. by conjugate
3. Then evaluate limit

ex. $\lim\limits_{x\to 2} \dfrac{x^2-4}{x-2} = \dfrac{(2)^2-4}{2-2} = \dfrac{0}{0}$ ← No good

$$\lim_{x\to 5} \frac{x^2-25}{x-5}$$
$$= \lim_{x\to 5} \frac{(x+5)\cancel{(x-5)}}{\cancel{x-5}}$$
$$= \lim_{x\to 5} x+5$$

Check every step if you can plug it in!

$$= 5+5$$
$$= \boxed{10}$$

extra example

ex. $\lim\limits_{x\to 2} \dfrac{x^2+x-6}{x-2} =$

$\lim\limits_{x\to 2} x+3 = 2+3 = \boxed{5}$

$a+b \to a-b$

mult. by conjugate

(See with roots / complex #5)

ex. $\lim\limits_{x\to 9} \dfrac{\sqrt{x}-3}{(x-9)}\left(\dfrac{\sqrt{x}+3}{\sqrt{x}+3}\right)$

$$= \lim_{x\to 9} \frac{x+\cancel{3\sqrt{x}}-\cancel{3\sqrt{x}}-9}{(x-9)(\sqrt{x}+3)}$$
$$= \lim_{x\to 9} \frac{\cancel{x-9}}{\cancel{(x-9)}(\sqrt{x}+3)}$$
$$= \lim_{x\to 9} \frac{1}{\sqrt{x}+3}$$
$$= \frac{1}{\sqrt{9}+3}$$
$$= \frac{1}{3+3} = \boxed{\frac{1}{6}}$$

ex. Calculate the limit if given
$\lim_{x\to 2} f(x) = 3$ & $\lim_{x\to 2} g(x) = 4$

covered on pg 4

Find $\lim_{x\to 2}\left[f(x) + 5g(x)\right]$

$= \lim_{x\to 2} f(x) + 5\lim_{x\to 2} g(x)$

$= 3 + 5(4)$

$= \boxed{23}$

Limits at infinity

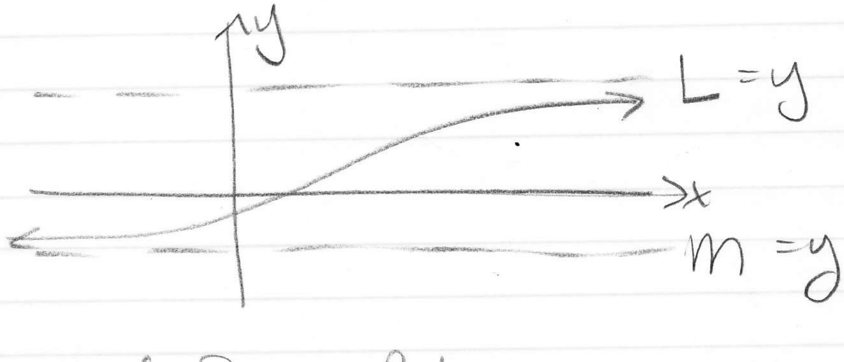
Long term behavior of graph

If f(x) approaches L as x increases without bound, then

$$\lim_{x\to\infty} f(x) = L$$

Similarly $\lim_{x\to -\infty} f(x) = M$

when f(x) approaches M as x decreases without bound.



Reciprocal Power Rules

If A & k are constants & k > 0,

$\lim_{x\to\infty} \frac{A}{x^k} = 0$ $\lim_{x\to -\infty} \frac{A}{x^k} = 0$

$\frac{\#}{BIG}$ = SMALL

$\frac{\#}{\infty} = 0$

Strategy to calculate limits at ∞.
(1) Divide each term by the highest power of x that appears in the denom
(2) Use reciprocal power rules so that terms go to zero.

ex. $\lim\limits_{x\to\infty} \dfrac{1-3x^3}{2x^3-6x+2}$

$= \lim\limits_{x\to\infty} \dfrac{\cancel{\frac{1}{x^3}}^{0} - \frac{3\cancel{x^3}}{\cancel{x^3}}}{\frac{2\cancel{x^3}}{\cancel{x^3}} - \cancel{\frac{6x}{x^3}}^{0} + \cancel{\frac{2}{x^3}}^{0}}$

$= \dfrac{0+3}{2+0} = \boxed{\dfrac{3}{2}}$

here

Talk about precalc

ex. $\lim\limits_{x\to\infty} \dfrac{3x^2-6x+2}{2x-9}$

$= \lim\limits_{x\to\infty} \dfrac{\frac{3x^2}{x} - \frac{6x}{x} + \frac{2}{x}}{\frac{2x}{x} - \frac{9}{x}} = \lim\limits_{x\to\infty} \dfrac{3x-6+\cancel{\frac{2}{x}}^{0}}{2-\cancel{\frac{9}{x}}^{0}} = \lim\limits_{x\to\infty} \dfrac{3x-6}{2} = \boxed{\infty}$

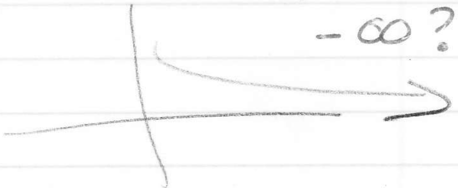
ex. $\lim\limits_{x\to\infty} \dfrac{x^2+x-5}{1-2x-x^3}$

$= \lim\limits_{x\to\infty} \dfrac{\frac{x^2}{x^3} + \frac{x}{x^3} - \frac{5}{x^3}}{\cancel{\frac{1}{x^3}}^{0} - \cancel{\frac{2x}{x^3}}^{0} - \frac{x^3}{x^3}}$

$= \lim\limits_{x\to\infty} \dfrac{\frac{1}{x} + \frac{1}{x^2} - \frac{5}{x^2}}{-1}$

$= \dfrac{0}{-1} = \boxed{0}$

ex. What about $-\infty$? $\to -\infty$



Pre calc Review

Rational Functions / Asymptote

Given $\frac{x^m}{x^n}$, if

* $m > n$ No horizontal asymptotes

  ex. $\frac{x^5}{x^4} = x$

* $m = n$

  ex. $\frac{a_1 x^3}{a_2 x^3} \Rightarrow$ horizontal asymptote @ $y = \frac{a_1}{a_2}$!

* $n > m$ There is a horiz. asymp. at $y=0$.

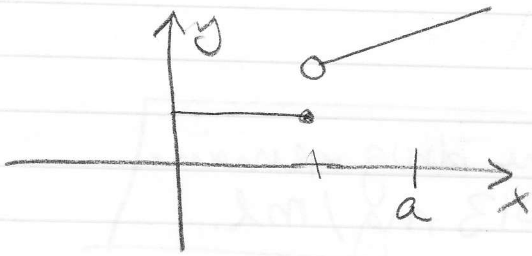
  ex. $\frac{1}{x}$

11am & 1pm

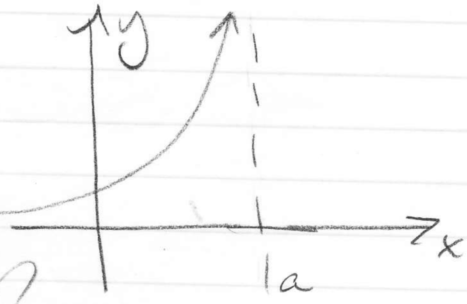
didn't cover

ex Find $\lim\limits_{x \to a} f(x)$ if it exists

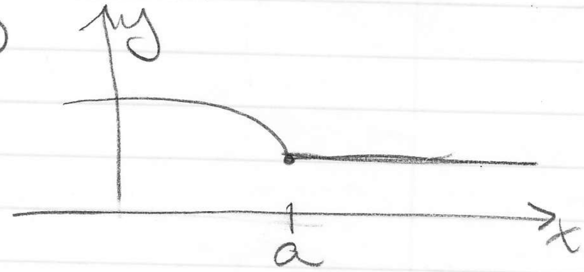
a)



b)



c)



?

if time!

Applications

ex. The concentration of drug in a patients blood stream t hrs after injection is C(t) milligrams per milliliter where

$$C(t) = \frac{0.4}{t^{1.2}+1} + 0.013$$

Concentration after 1 hr?

a) What is the initial concentration of the drug (t = 0)?

$C(0) = 0.013$

The concentration is 0.013 mg/ml

b) By how much does the concentration change during the 5th hr? Increase/decrease?

$C(5) - \underset{\text{Bigger}}{C(4)} = \frac{0.4}{5^{1.2}+1} + \cancel{0.013} - \frac{0.4}{4^{1.2}+1} - \cancel{0.013}$

$= -0.01307$ decrease (sentence)

c) What is the residual concentration? Amt in blood stream in long run $\lim\limits_{t \to \infty} \frac{0.4}{t^{1.2}+1} + 0.013 = 0.4 \lim\limits_{t \to \infty} \frac{1}{t^{1.2}+1} + \lim\limits_{t \to \infty} 0.013 =$ →

$$= \lim_{t \to \infty} 0.4 \left(\frac{\frac{1}{t^{1.2}} \to 0}{\frac{t^{1.2}}{t^{1.2}} \to 1 + \frac{1}{t^{1.2}} \to 0} \right) + 0.013$$

$$0.4(0) + 0.013$$

$$= 0.013$$

$\boxed{\text{In the long run, the drug remains in system with } 0.013 \text{ mg/ml.}}$