Section 1.1 - Functions

Motivation: In many real world situations things are related in certain ways. For instance, the amount of pollution in a certain area may depend on the number of cars on the road, temperature in a room might depend on the number of people in the room. Rate that soup cools depends on room temp. These situations can be represented as functions.

Function Defn
A rule that assigns to each object in a set A exactly one object in a set B.
Set A = Domain (Real #s)
Set B = Range

Usually

\[ \text{map} \quad A \rightarrow B \]

Function Notation
\[ y = f(x) \]
Plug in an x value to get a y value out
Function Evaluation

ex) Find $f(-3)$ if $f(x) = x^2 + 4x - 2$

\[ f(-3) = (-3)^2 + 4(-3) - 2 \]
\[ = (-3)^2 + 4(-3) - 2 \]
Notation
\[ v = -3 - 2 \]
\[ v = -5 \]
Other representation
\[ \Rightarrow x = -3, \ y = -5 \]
\[ (-3, -5) \]

Evaluate

ex) \[ g(t) = (2t - 8)^{1/2} \]
\[ g(t) = \sqrt{2(t) - 8} \]
\[ g(9) = \sqrt{2(9) - 8} \]
\[ = \sqrt{18 - 8} \]
\[ = \sqrt{10} \]
\[ g(4) = \sqrt{2(4) - 8} \]
\[ = 0 \]
\[ g(0) = \sqrt{2(0) - 8} \]
\[ = \sqrt{-8} \]
\[ = \text{undefined (imaginary)} \]

1pm - Day 11
ex) \[ h(x) = \begin{cases} 
-2x + 4 & \text{if } x \leq 1 \\
 x^2 + 1 & \text{if } x > 1 
\end{cases} \]
\[ h(3) = (3)^2 + 1 \]
\[ = 10 \]
\[ h(1) = -2(1) + 4 \]
\[ = 2 \]
Natural Domain
All x values for which the function is defined. Restrictions on domain.

Linear \( f(x) = mx + b \)  Domain = All real numbers \( \mathbb{R} \) or \( (-\infty, \infty) \)

Polyomials \( g(x) = ax^n + bx^{n-1} + \ldots + c \)  Domain = \( \mathbb{R} \)

Square Roots
Negative numbers do not have real square roots.

Radical
Radical must be positive or zero \( \sqrt{\text{radicand}} \geq 0 \)

ex. \( f(x) = \sqrt{2x + 5} \)

Domain  Radicand \( \geq 0 \)

\(-2x + 5 \geq 0 \)
\[-2x \geq -5 \]
\[x \leq \frac{5}{2} \]

Rational Functions
\( \frac{Q(x)}{R(x)} \) where \( Q(x) \neq R(x) \) are polynomials

Restriction on domain: Denom cannot equal zero \( \frac{\text{numerator}}{\text{denominator}} \)

ex. \( g(x) = \frac{3}{4x - 5} \)

Denom \( \neq 0 \)
\[4x - 5 \neq 0 \]
\[4x \neq 5 \]
\[x \neq \frac{5}{4} \]

ex. \( f(x) = \frac{\sqrt{\text{radicand}}}{\sqrt{2x + 5}} \)

Domain \( \text{radicand} > 0 \)

\(3 \times 1 \times x \neq \frac{5}{4} \)  \( (-\infty, 5) \cup (\frac{5}{4}, \infty) \)
Application in Economics

Functions dealing with marketing

Demand - price $p = D(x)$ that is charged for each unit of commodity if $x$ units are to be sold.

Supply - price $p = S(x)$ at which producers are willing to supply $x$ units to market.

Revenue - $R(x)$ is revenue from selling $x$ units

$$P(x) = (# \text{ of units sold}) \cdot (\text{price per item})$$

$$= x \cdot p(x)$$

Cost - $C(x)$ is the cost of producing $x$ units.

Profit - $P(x)$ is profit from selling $x$ units of the commodity.

$$P(x) = \text{Revenue} - \text{Costs}$$

$$= R(x) - C(x)$$

$$= x \cdot p(x) - C(x)$$

(Higher the unit price, the fewer the # of units demanded & Vice versa)

Example: Given $p = D(x)$ & $c(x)$ find

a) The revenue $R(x)$ & profit $P(x)$

b) All values of $x$ for which production of the commodity is profitable.
**LEAVE OUT**

ex. cont.

\[ D(x) = -0.5x + 39 = p(x) \]

\[ C(x) = 1.5x^2 + 9.2x + 67 \]

\[ R(x) = x \cdot p(x) = x(-0.5x + 39) \]

\[ R(x) = -0.5x^2 + 39x \]

\[ P(x) = R(x) - C(x) = -0.5x^2 + 39x - (1.5x^2 + 9.2x + 67) \]

\[ P(x) = -2x^2 + 29.8x - 67 \]

\[ a = -2 \quad b = 29.8 \quad c = -67 \]

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

\[ \quad \quad = \frac{-29.8 \pm \sqrt{29.8^2 - 4(-2)(-67)}}{2(-2)} \]

\[ = 2.76 \quad + \quad 12.14 \]

Profit > 0 when

\[ 2.76 \leq x \leq 12.14 \]

\[ (2.76, 12.14) \]
Function Composition

\[ g \quad \overset{\text{output}}{\longrightarrow} \quad g(x) \quad \overset{\text{input}}{\longrightarrow} \quad f \quad \overset{\text{output}}{\longrightarrow} \quad f(g(x)) \]

Notation

\((f \circ g)(x) = f(g(x)) \quad (g \circ f)(x) = g(f(x))\)

"f of g of x" inner

NOT times

In general \(f(g(x)) \neq g(f(x))\)

ex. Find \(f(g(x))\) if \(f(u) = u^2 + 4\), \(g(x) = x - 1\)

\[ f(g(x)) = f\left( x - 1 \right) \]
\[ = (x - 1)^2 + 4 \]
\[ = (x - 1)(x - 1) + 4 \]
\[ = x^2 - 2x + 1 + 4 \]
\[ = x^2 - 2x + 5 \] composite function

ex. Find \(f(5x^2 + 4x)\)

if \(f(x) = 2x + 3\)

\[ f(5x^2 + 4x) = 2(5x^2 + 4x) + 3 \]
\[ = 10x^2 + 8x + 3 \]

ex. If \(f(x) = \sqrt{x+1}\), find \(h(x)\) & \(g(w)\) so that \(f(x) = g(h(x))\)

ex. If \(f(x) = \frac{1}{x^2 + 1}\), find \(h(x)\) & \(g(w)\) so that \(f(x) = g(h(x))\)

Option 1: \(h(x) = x^2 + 1\)

\[ g(w) = \frac{1}{w} \]

Option 2: \(h(x) = \frac{1}{u}\)

\[ g(w) = \frac{1}{u + 1} \]

Option 3: \(h(x) = x^2 + 1\)

\[ g(u) = u^{-1} \]
Difference Quotient (Expression)
\[ \frac{f(x+h) - f(x)}{h} \]
*Important to define derivative
*Fundamental concept of calculus

Ex. Find the difference quotient for \( f(x) = 3 - x^2 \)

\[
f(x) = 3 - x^2
\]

\[
f(x+h) = 3 - (x+h)^2
\]
\[
= 3 - (x^2 + 2xh + h^2)
\]
\[
= 3 - x^2 - 2xh - h^2
\]

\[
f(x+h) - f(x) = \frac{3 - x^2 - 2xh - h^2 - (3 - x^2)}{h}
\]
\[
= \frac{-2xh - h^2}{h}
\]
\[
= h(-2x-h)
\]

Can't cancel parts of a sum
\[
\frac{8}{5} = \frac{5+3}{5} = 3
\]

No!

Skip application

#66, #79 (Similar to #80)

74. Manufacturing Cost
\[ C(q) = 1200 + 900q \]
\[ q(t) = 4t + 6 \]

a) Express cost as a function of time \( (C(t)) \)
\[ (C(t)) = 625t^2 + 25t + 900 \]

b) \( C(3) = 6600 \) a) \( t = 4 \)

25t^2 + 40t + 100 = 0
\( a = 25 \), \( b = 40 \), \( c = 100 \)

\( x = -\frac{b}{2a} \)
\( x = -\frac{40}{50} = -0.8 \)

\( x = 0 \)
Another example of finding difference quotient:

Find the Difference Quotient given

\[ f(x) = \frac{3}{x+1} \]

The Difference Quotient

\[ \frac{f(x+h) - f(x)}{h} = \frac{3}{(x+h)+1} - \frac{3}{x+1} \]

\[ = \frac{3}{x+h+1} - \frac{3}{x+1} \]

Now get a common denominator & subtract

\[ = \frac{3(x+1) - 3(x+h+1)}{(x+h+1)(x+1)} \]

\[ = \frac{3x+3 - 3x - 3h - 3}{(x+h+1)(x+1)} \]

\[ = \frac{3x+3 - 3x - 3h - 3}{(x+h+1)(x+1)} \cdot \frac{1}{h} \]

\[ = \frac{-3h}{(x+h+1)(x+1) \cdot h} \]

\[ = \frac{-3}{(x+h+1)(x+1)} \]

Don't worry about "wasting" paper.
Write big & show steps!!
Find the difference quotient:

\[ f(x) = \frac{3}{x} \]

\[
\frac{f(x+h) - f(x)}{h} = \frac{\frac{3}{x+h} - \frac{3}{x}}{h}
\]

\[
= \frac{\frac{3x - 3(x+h)}{x(x+h)}}{h}
\]

\[
= \frac{3}{x(x+h)} \cdot \frac{1}{h}
\]

\[
= \frac{3}{x(x+h)}
\]
\[ f(x) = \sqrt{x} \]

\[ g(x) = -5x + 1 \]

\[ f(g(x)) = f(-5x + 1) \]

\[ = \sqrt{-5x + 1} \]

\[ g(f(x)) = g(\sqrt{x}) \]

\[ = -5(\sqrt{x}) + 1 \]

\[ = -5\sqrt{x} + 1 \]

\[ (\sqrt{-5x + 1})^2 = (5\sqrt{x} + 1)^2 \]

\[-5x + 1 = 25\sqrt{x}\sqrt{x} + 10\sqrt{x} + 1 \]

\[-5x + 1 = 25x + 10\sqrt{x} \]

\[-30x - 1 = 10\sqrt{x} \]

\[-30x - 1 = 10\sqrt{x} \]

\[ (3x)^2 = (5x)^2 \]

\[ 9x^2 = x \]

\[ 9x^2 - x = 0 \]

\[ x(9x - 1) = 0 \]

\[ x = 0 \]

\[ x = \frac{1}{9} \]

\[ \sqrt{-5\left(\frac{1}{9}\right) + 1} \]

\[ = -5\sqrt{\frac{1}{9}} + 1 \]

\[ = -\frac{5}{3} + 1 \]

\[ \sqrt{\frac{4}{9}} = -\frac{5}{3} + \frac{3}{3} \]

\[ \frac{2}{3} = -\frac{2}{3} \]

No