A Mathematical Model of Bacterial Aerotaxis

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Outline

- Chemotaxis and Aerotaxis

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- Pattern formation in aerotaxis experiments (Zhulin et al. 1996)

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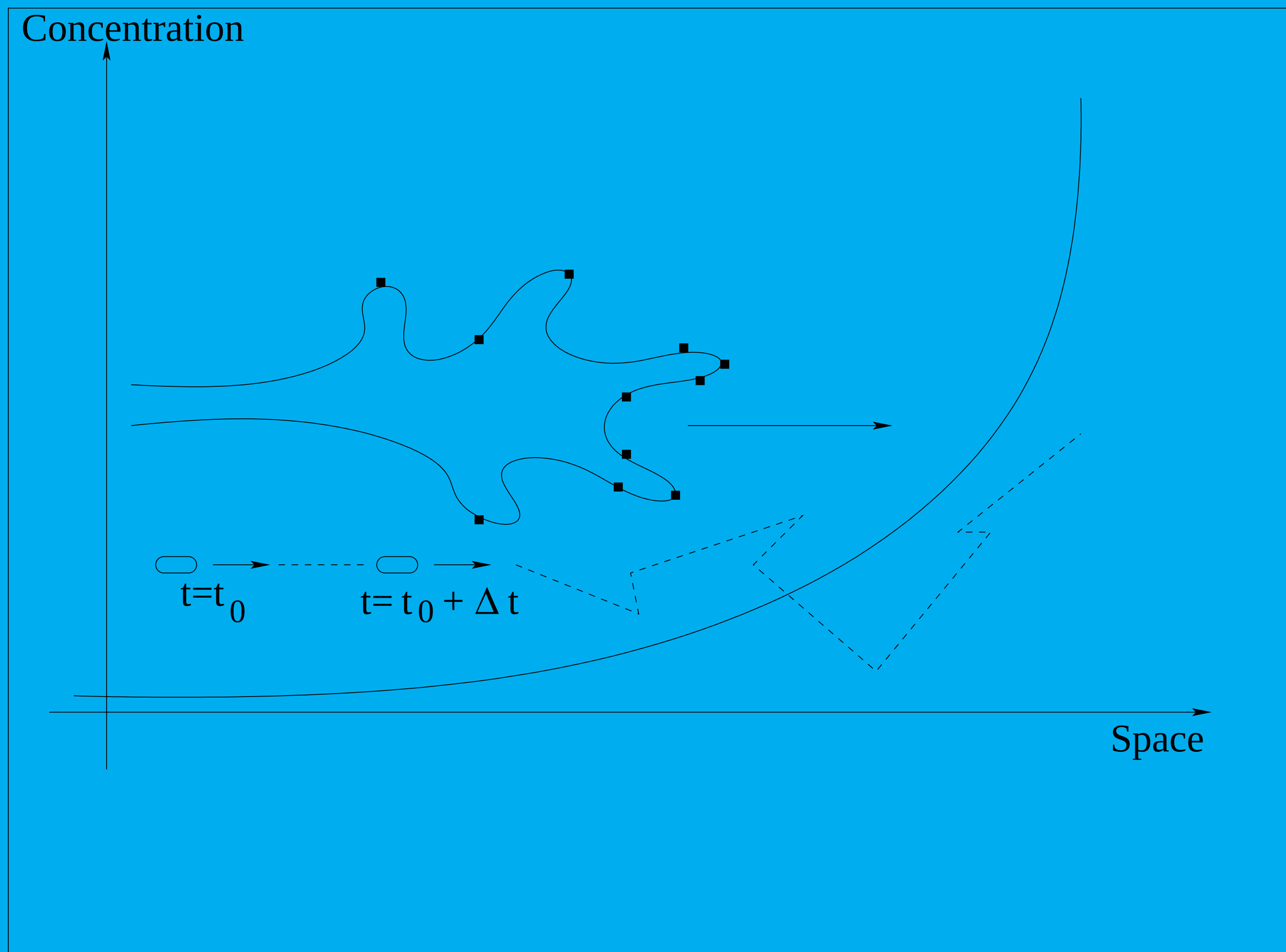
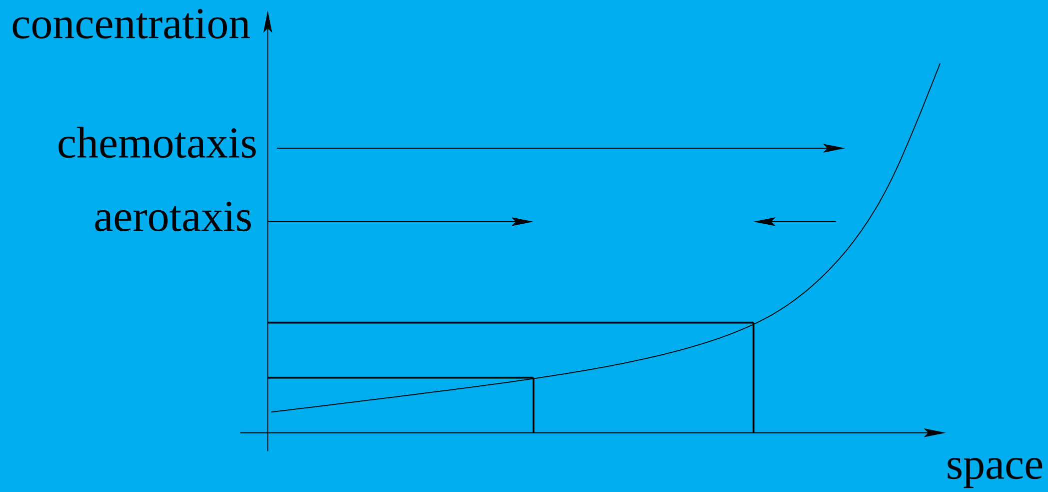
- Our model
- Results
- Significance and interpretation of our results

Conventional Chemotaxis



Chemotaxis and Aerotaxis



Chemotaxis

many attractants and repellents

effector need not be metabolized

external effector concentrations are monitored

signaling: Tar, Tsr, Trg, Tap

Aerotaxis

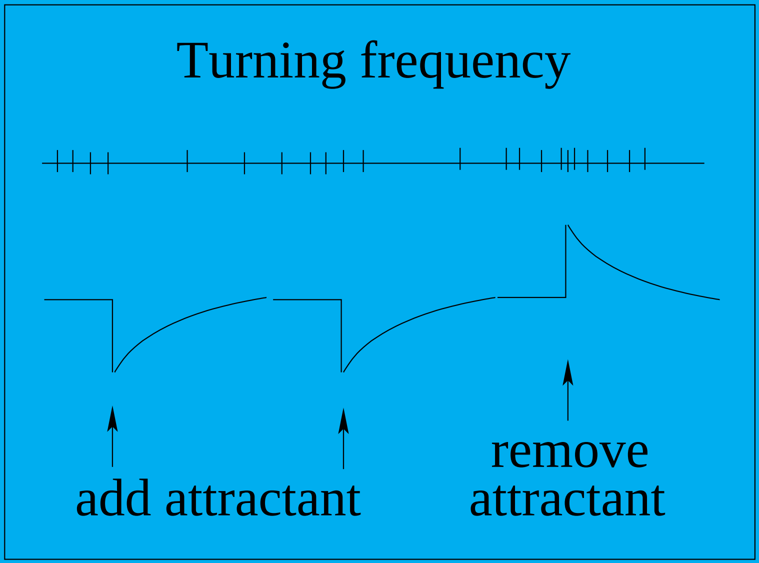
oxygen is attractant and repellent

oxygen is metabolized

energy taxis − internal state monitored

signaling: Tsr, Aer

Adaptation in conventional chemotaxis

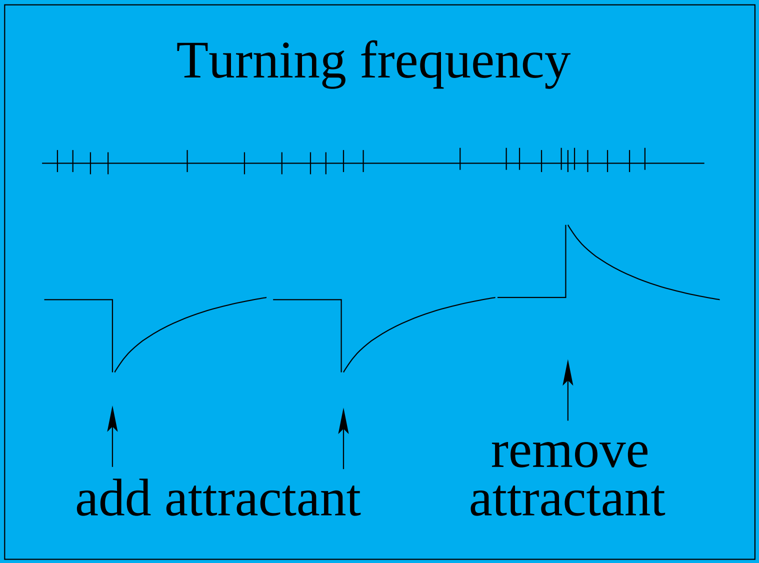


(Figure based on Bray, 1992)

Fast excitation (phosphorylation)

Slow adaptation (methylation)

Adaptation in conventional chemotaxis



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observable

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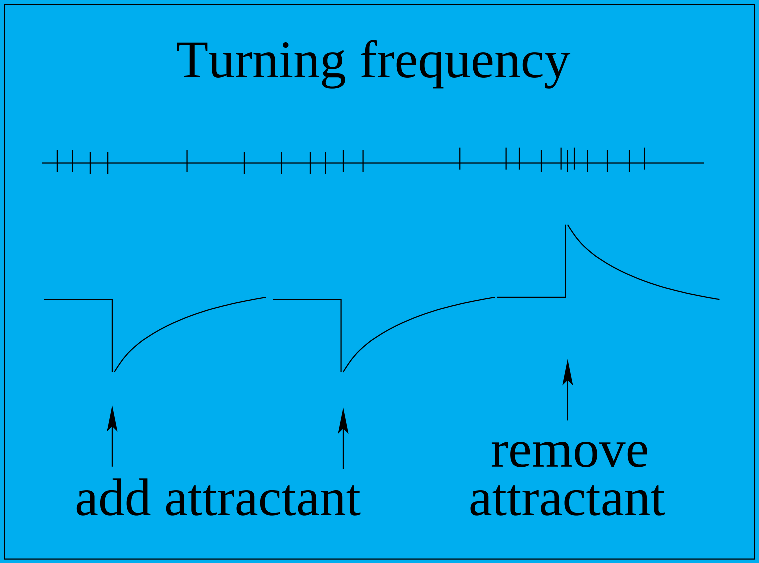

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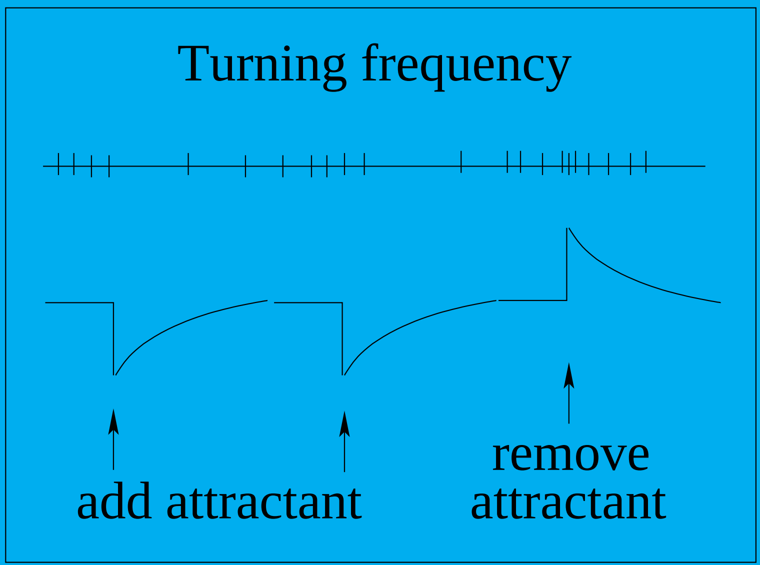

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allows sensing in a wide range of concentrations

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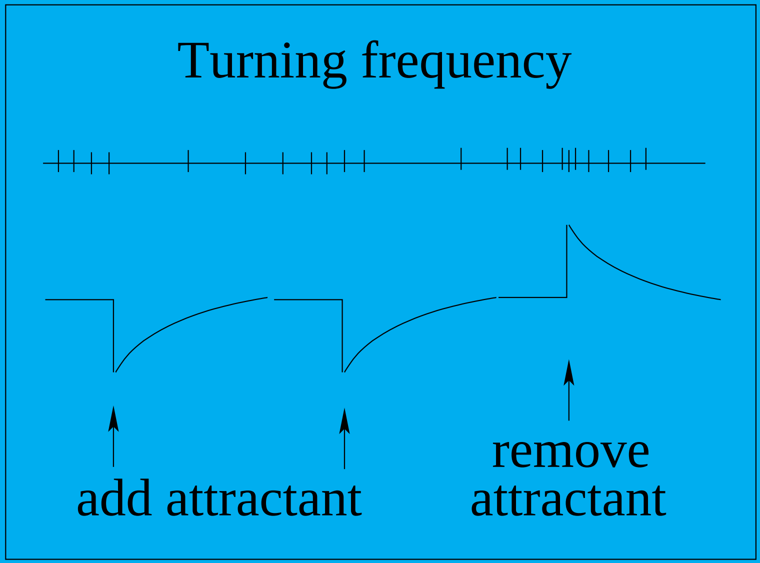

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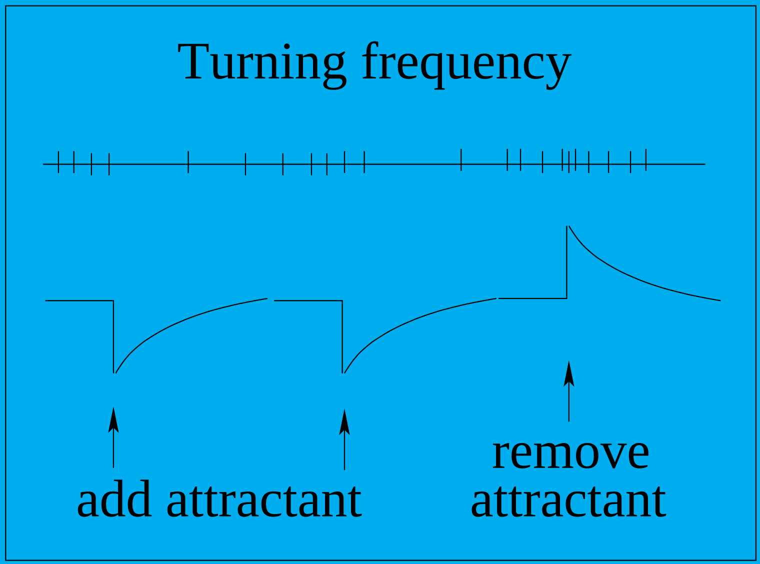

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Fast excitation (phosphorylation)

Slow adaptation (methylation)

observable

serves as memory

allows sensing in a wide range
of concentrations

methylation based

slow (1-3 seconds)

Keller-Segel chemotaxis model

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$$\frac{\partial b^+}{\partial t} + v\frac{\partial b^+}{\partial x} = \sigma^- b^- - \sigma^+ b^+ \quad (1)$$

$$\frac{\partial b^-}{\partial t} - v\frac{\partial b^-}{\partial x} = \sigma^+ b^+ - \sigma^- b^- \quad (2)$$

Keller-Segel chemotaxis model

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$$\frac{\partial b^-}{\partial t} - v\frac{\partial b^-}{\partial x} = \sigma^+ b^+ - \sigma^- b^- \tag{2}$$

Let $\quad b = b^+ + b^-, \quad J(x,t) = v(b^+(x,t) - b^-(x,t)),$
$\sigma_0 = \frac{1}{2}(\sigma^+ + \sigma^-), \quad \Delta\sigma = \frac{1}{2}(\sigma^+ - \sigma^-).$

Keller-Segel chemotaxis model

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Let $\quad b = b^+ + b^-, \quad J(x,t) = v(b^+(x,t) - b^-(x,t)),$ $\sigma_0 = \frac{1}{2}(\sigma^+ + \sigma^-), \quad \Delta\sigma = \frac{1}{2}(\sigma^+ - \sigma^-).$ By making the substitutions and adding 1 and 2, we get the equations:

$$b_t = -J_x \tag{3}$$

$$\frac{1}{2\sigma_0} J_t + J = \frac{1}{2\sigma_0}(-v^2 b_x + 2\Delta\sigma v b) \tag{4}$$

In small spatial gradients we can solve for J:

$$J \approx \frac{1}{2\sigma_0}(-v^2\frac{\partial b}{\partial x} + 2\Delta\sigma v b)$$

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Leading to the Keller-Segel (1970) equation:

$$\frac{\partial b}{\partial t} \approx \frac{\partial}{\partial x}(\mu\frac{\partial b}{\partial x} - \chi b) \tag{5}$$

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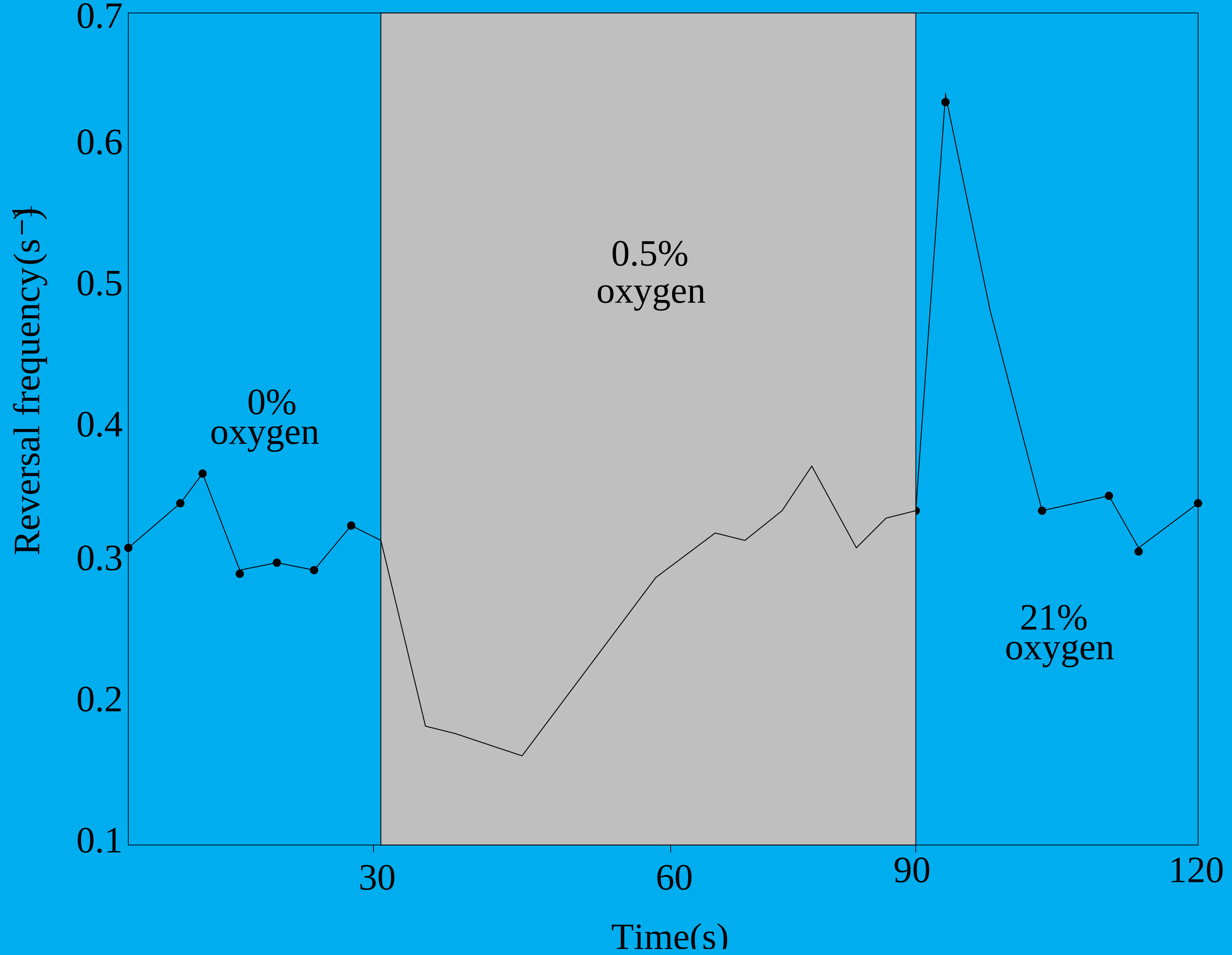
with

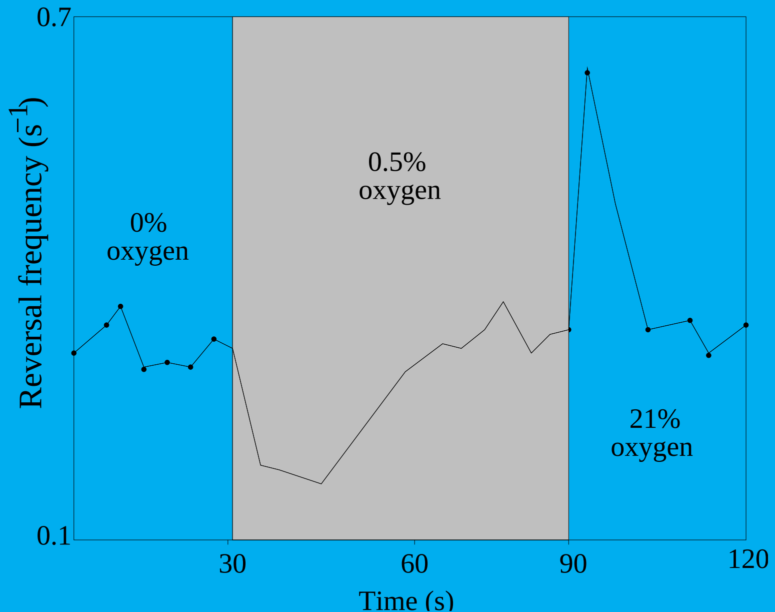
$$\mu = \frac{v^2}{2\sigma_0}$$
$$\chi = \frac{v\Delta\sigma}{\sigma_0}$$

(Grünbaum, 1999)

Temporal assay with *Azospirillum brazilense*



Membrane potential and turning frequency



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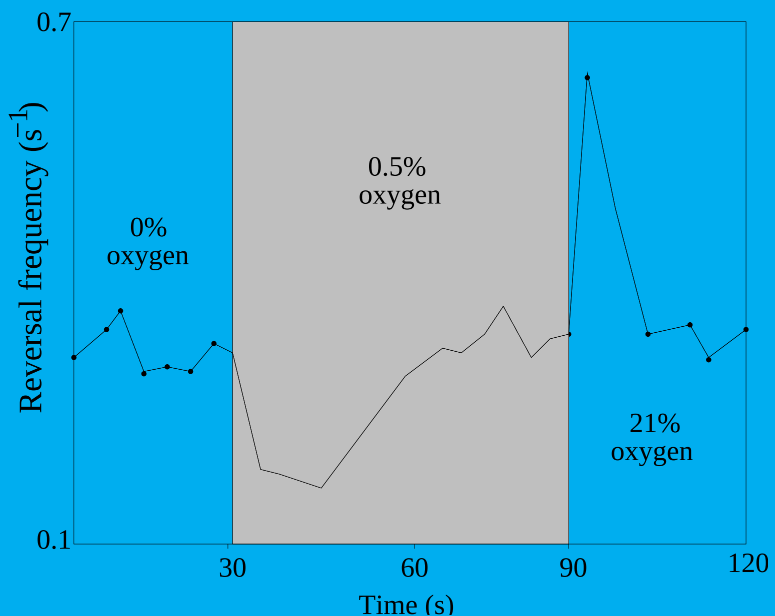

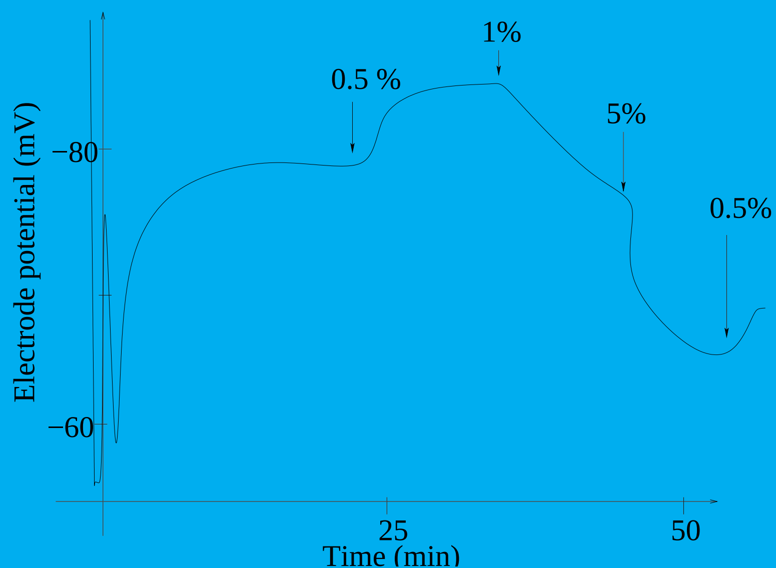

Spatial assay

Capillary size:

$50 \times 2 \times 0.1$mm

Oxygen concentration in band:

$0.3 - 0.5 \mu M$

Band width: 0.2 mm

Time of band formation:

50 sec - 3 min

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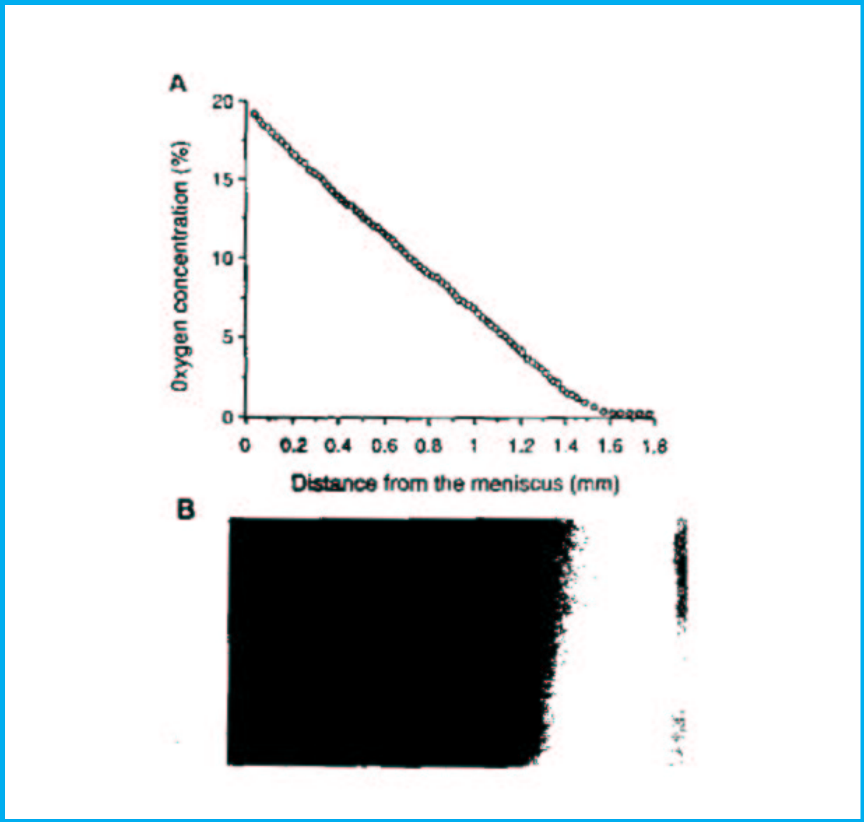
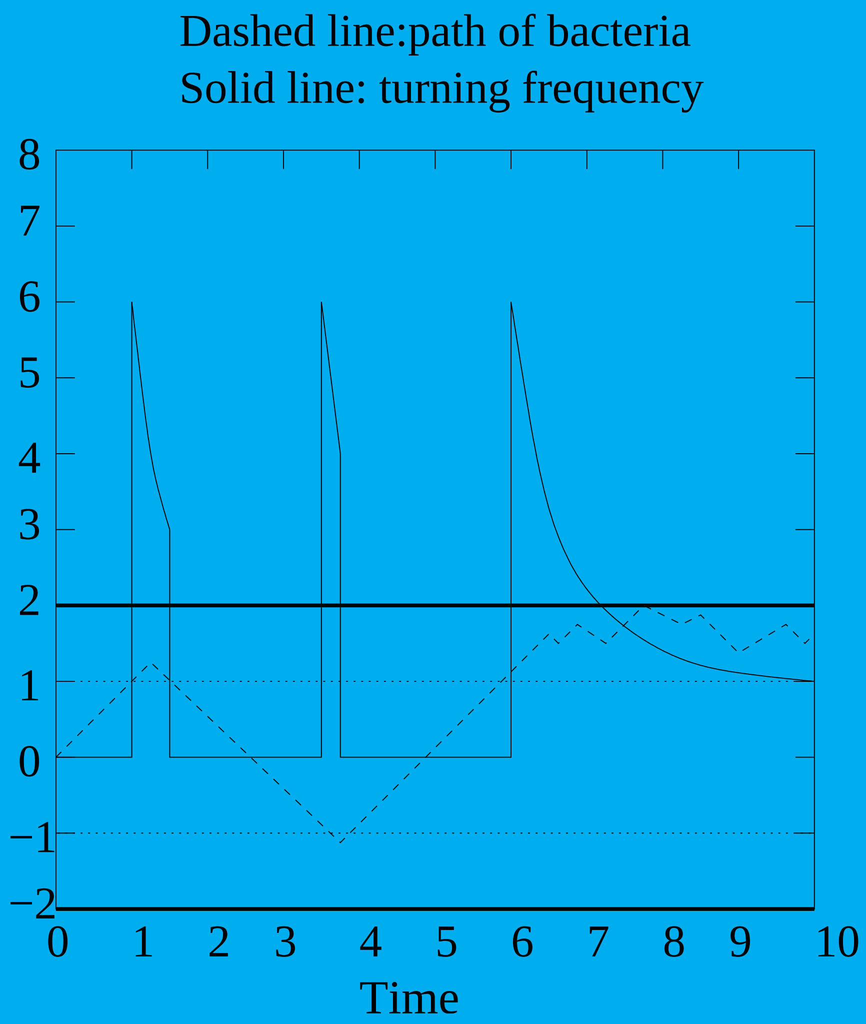


Monte-Carlo simulation



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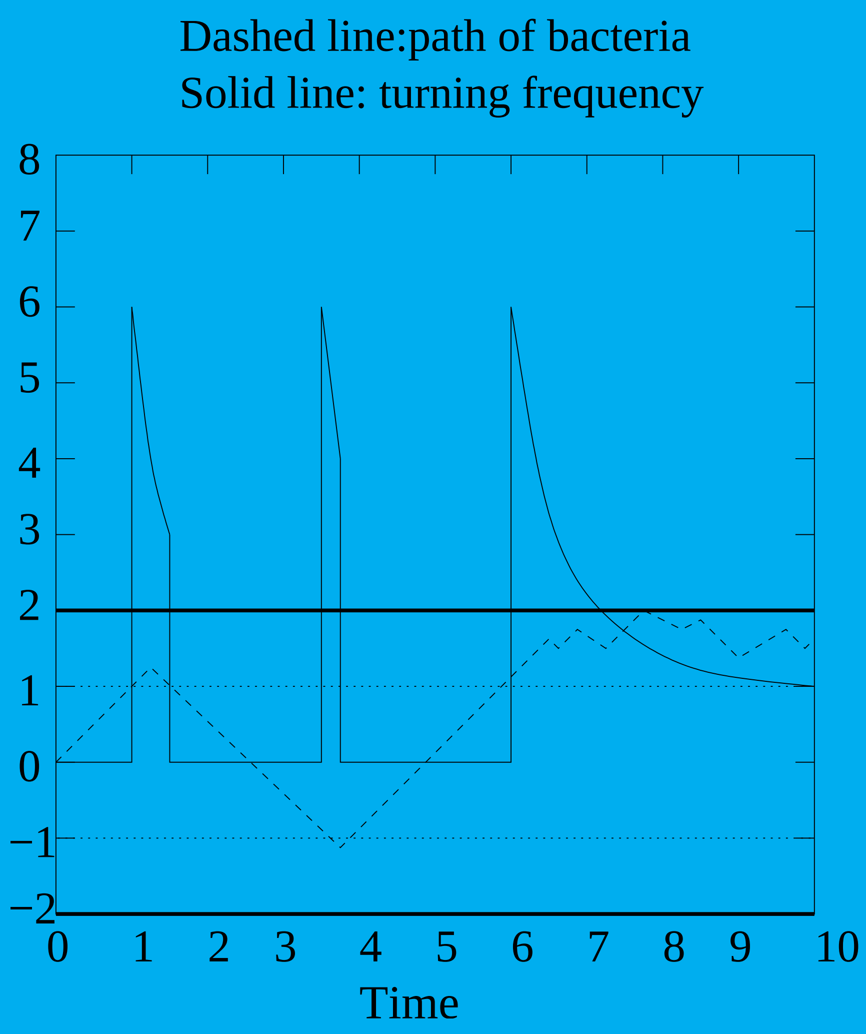

constant velocity, v

turning frequency
in band: $\sigma = 0$

cell leaves band
at random time, τ

outside the band
σ jumps to c

characteristic time
of adaptation: t_a

adaptation: $\sigma = ce^{-\frac{t-\tau}{t_a}}$

Modeling goals and assumptions

Goal: create a mathematical model which reproduces sharp aerotactic band formation in spatial assays with *Azospirillum brazilense*

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Assumptions:

- bacterial swimming is one-dimensional and it has constant velocity
- no bacterial reproduction
- oxygen concentration is fixed at the open end of the capillary
- no slow adaptation

Our model

$$\frac{\partial r}{\partial t} = -v\frac{\partial r}{\partial x} - f_{rl}r + f_{lr}l$$
$$\frac{\partial l}{\partial t} = v\frac{\partial l}{\partial x} + f_{rl}r - f_{lr}l$$

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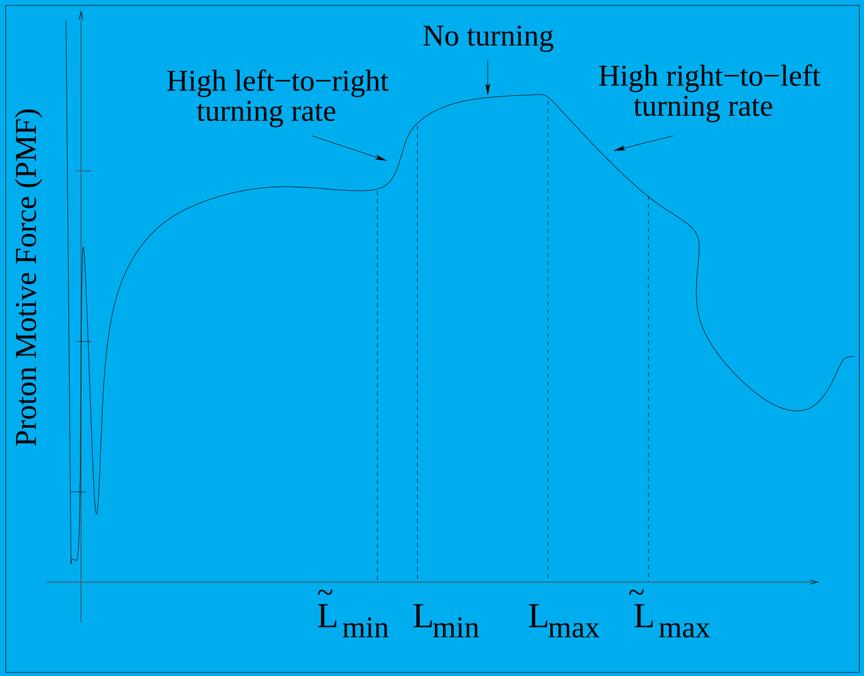
$$r(0,t) = l(0,t)$$
$$r(l,t) = l(l,t)$$
$$L(0,t) = L_0$$
$$\left.\frac{\partial L}{\partial x}\right|_{x=l} = 0$$

Turning rates



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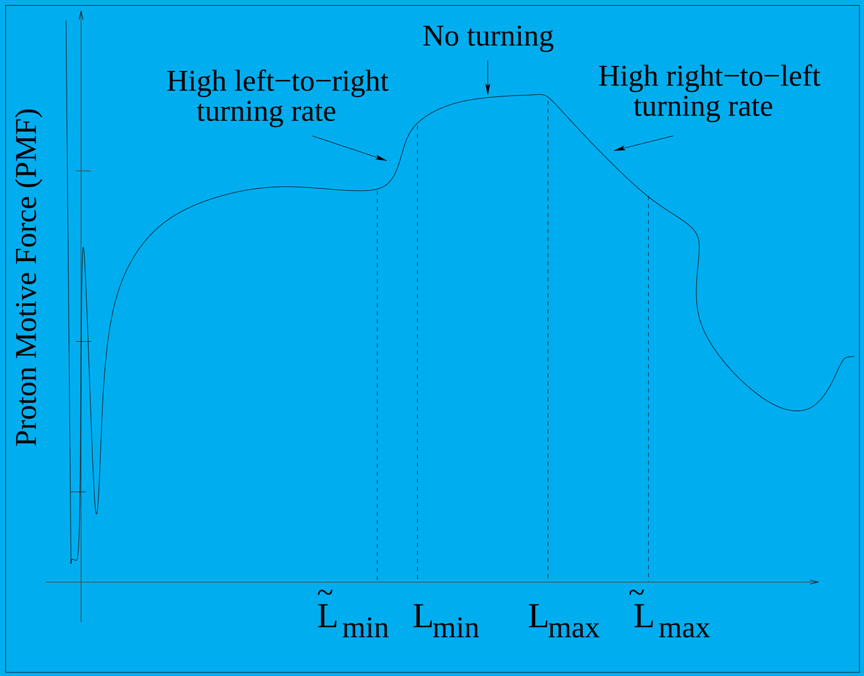

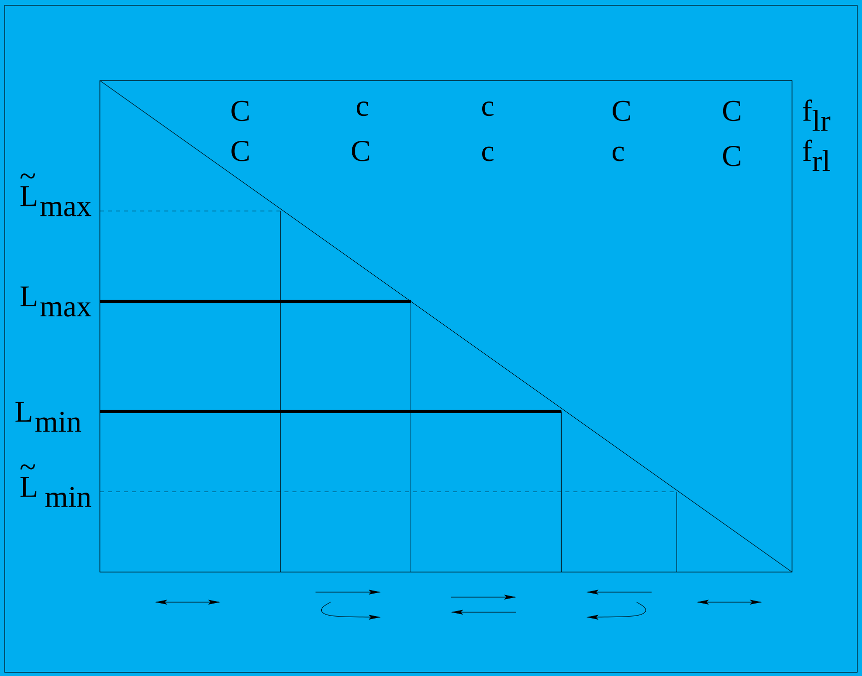

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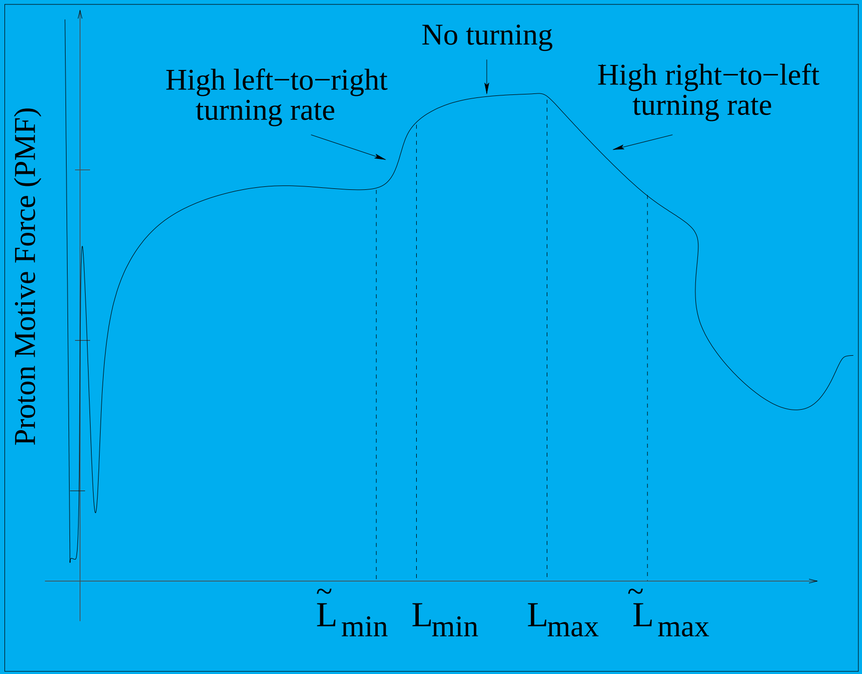

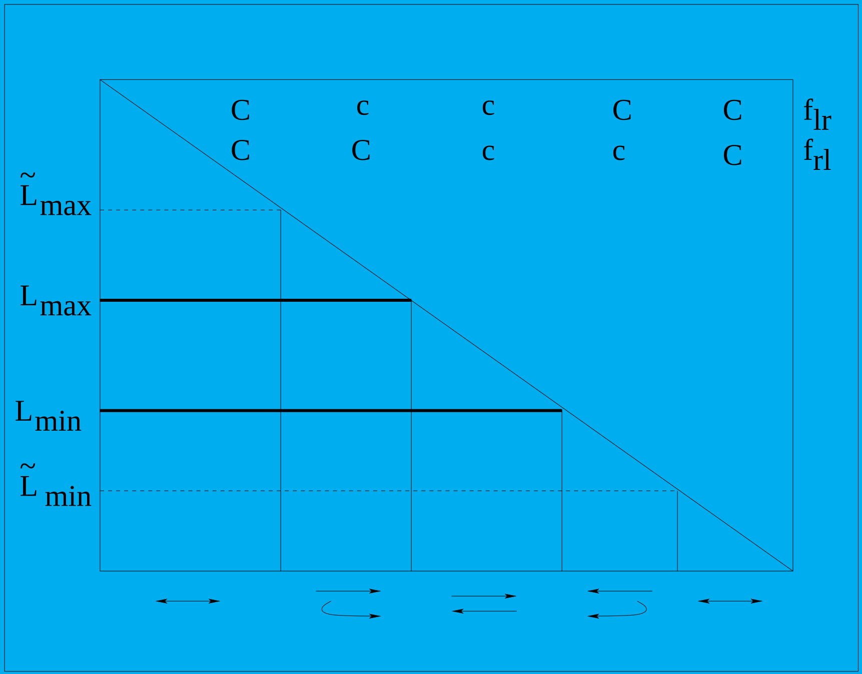

$$f_{rl} = \begin{cases} C, & L < \tilde{L}_{min} \\ c, & \tilde{L}_{min} < L < L_{min} \\ c, & L_{min} < L < L_{max} \\ C, & L_{max} < L < \tilde{L}_{max} \\ C, & \tilde{L}_{max} < L \end{cases}$$

Parameters and nondimensionalization

Measurement	Dim. quantity	Non-dim. value
Length	2 mm	1
Time	10 sec	1
Oxygen conc.	$1\ \frac{\mu M}{ml}$	1
Bacterial conc.	$2 \cdot 10^7 \frac{cells}{ml}$	1
Speed	$40 \frac{\mu m}{sec}$	0.2
Diffusion coeff.	$2 \cdot 10^{-9} \frac{m^2}{sec}$	0.01
Turning frequency	$1 sec^{-1}$	10
Rate of O_2 consump.	$3 \cdot 10^{-11} \frac{\mu M}{(cell)(sec)}$	$4 \cdot 10^{-3}$

Numerical results

Invasion of oxygen: $x \simeq \sqrt{Dt}$, Escape of bacteria: $x \simeq vt$

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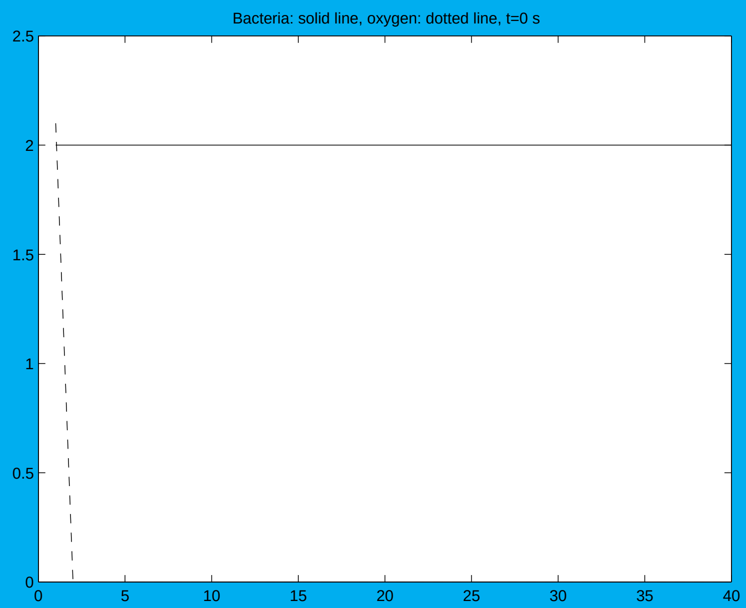

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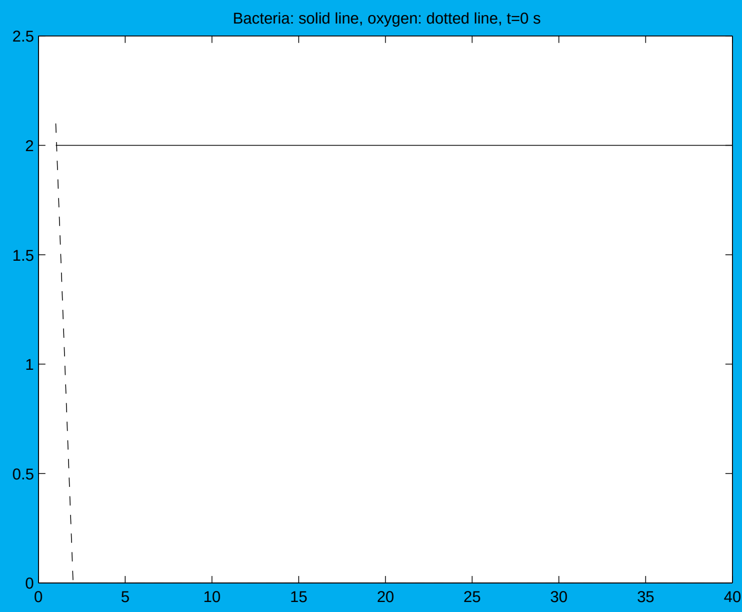

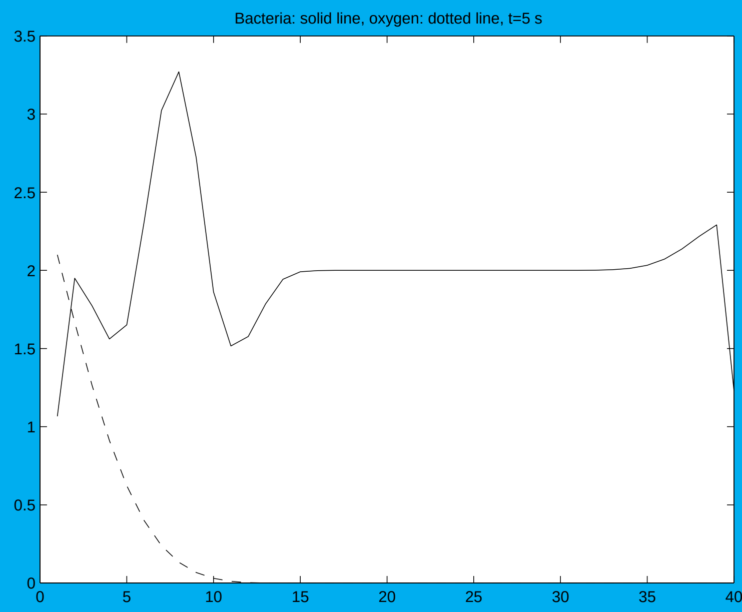

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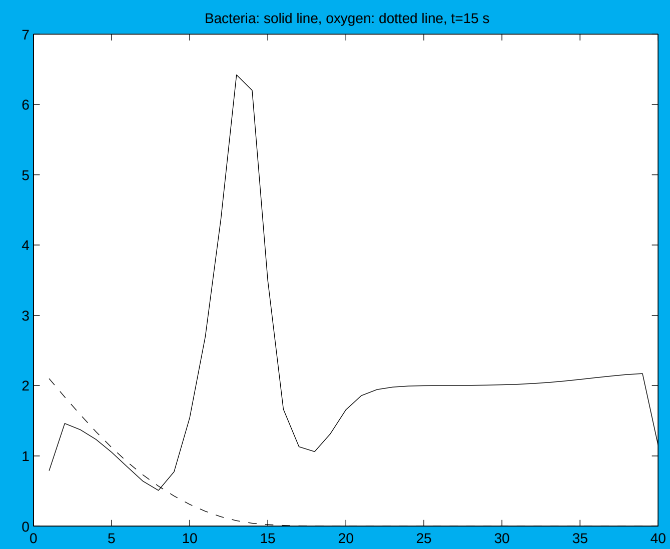
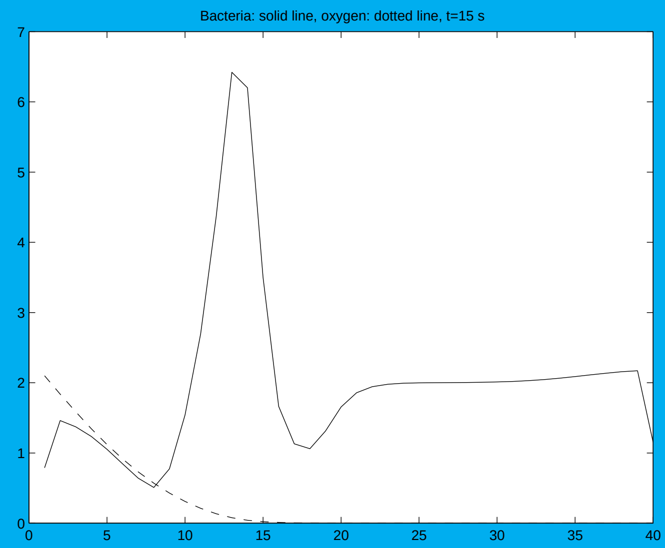


Bacteria: solid line, oxygen: dotted line, t=15 s
0
1
2
3
4
5
6
7
0
5
10
15
20
25
30
35
40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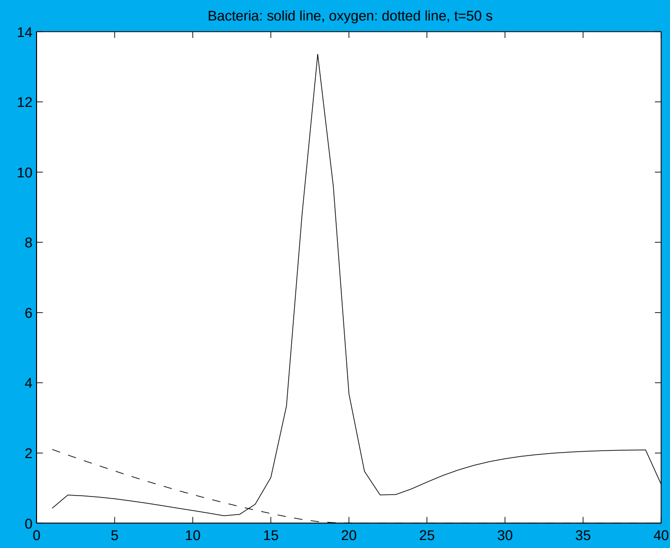
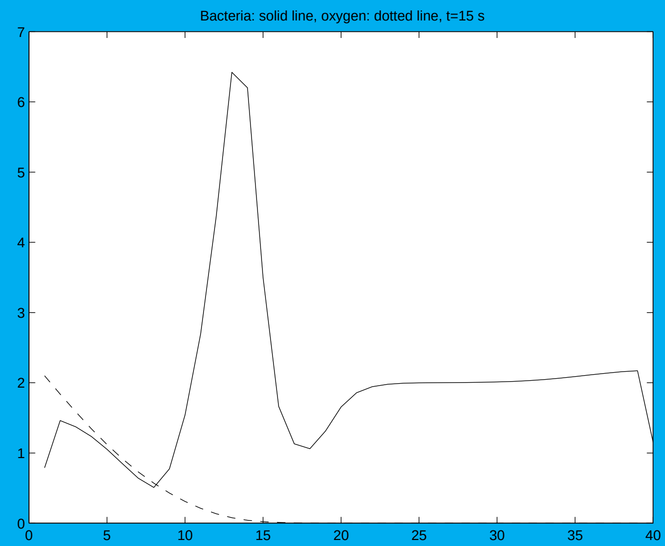

Bacteria: solid line, oxygen: dotted line, t=50 s
0
2
4
6
8
10
12
14
0
5
10
15
20
25
30
35
40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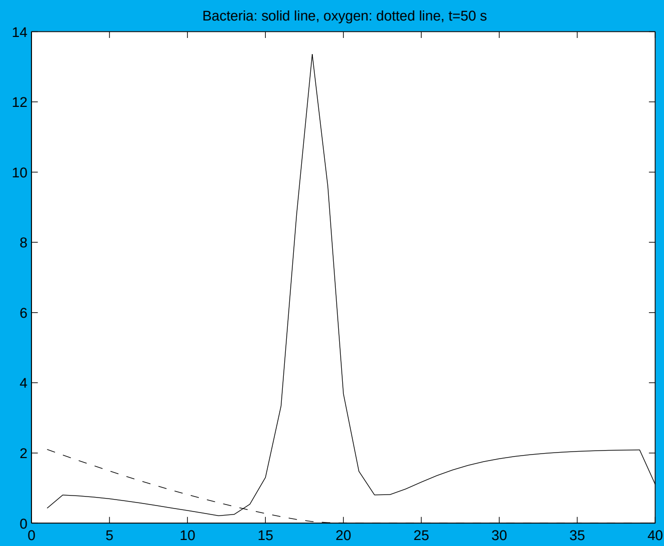
40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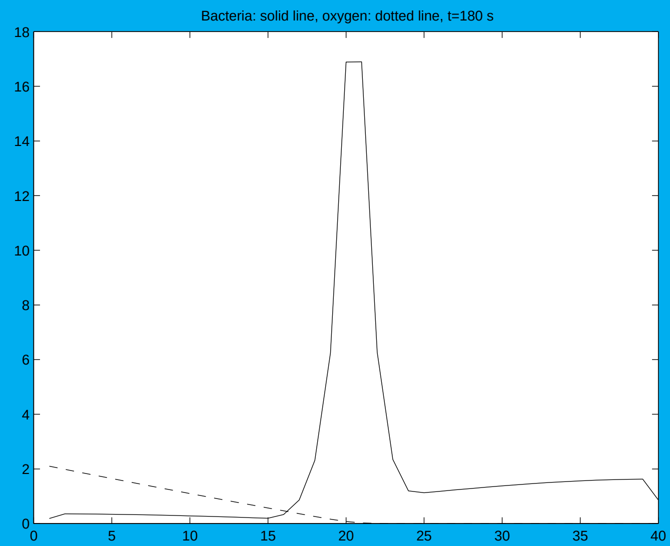
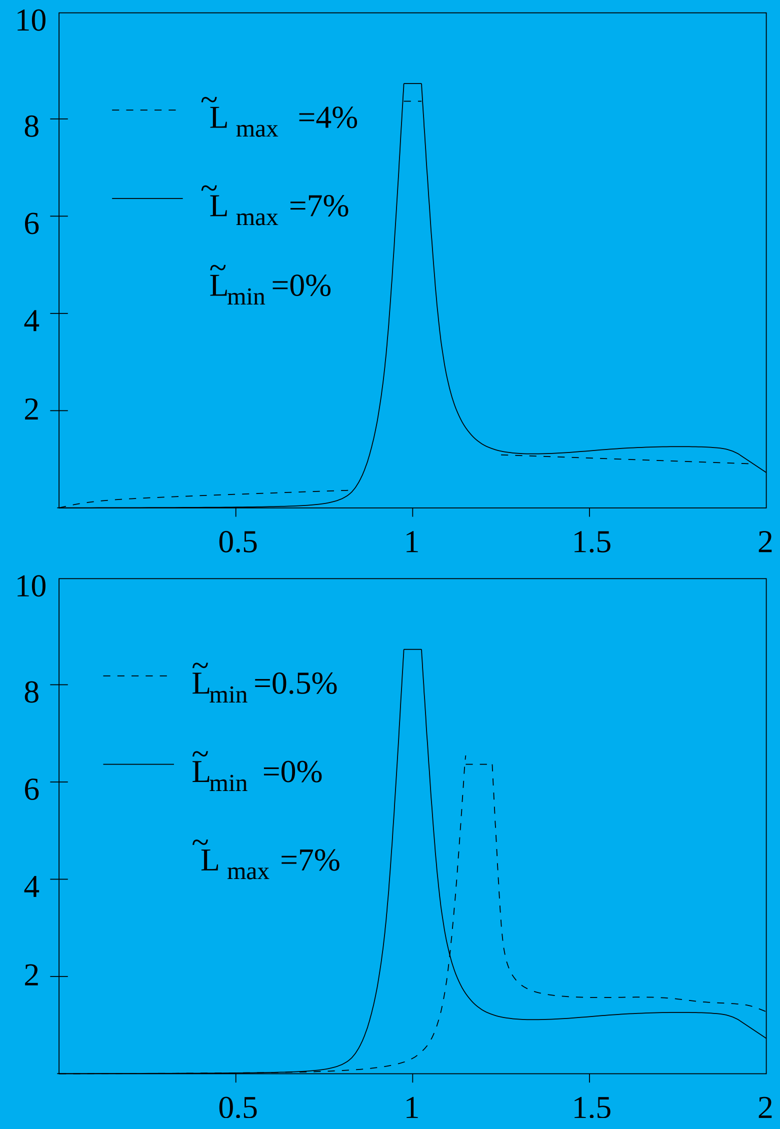
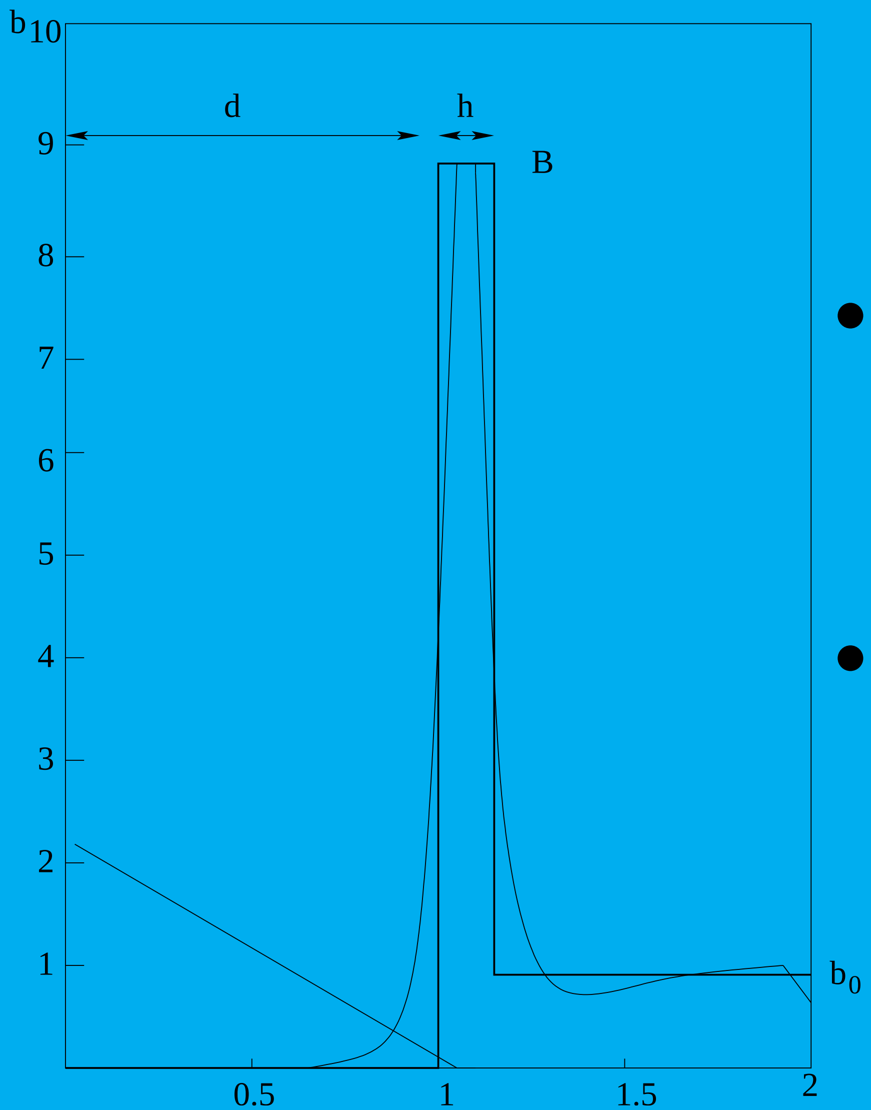
2
4
6
8
10
12
14
0
5
10
15
20
25
30
35
40


Bacteria: solid line, oxygen: dotted line, t=180 s
0
2
4
6
8
10
12
14
16
18
0
5
10
15
20
25
30
35
40

Numerical experiments with $\tilde{L}_{max}$ and $\tilde{L}_{min}$



Quasi steady state solution



- $d \approx \sqrt{\frac{L_0}{kb_0}}$
  $d \approx 0.8$ mm for $L_0 = 0.2$ and
  $d \approx 1.7$ mm for $L_0 = 1$

- $h \approx \frac{2L_{max}}{kb_0}\sqrt{\frac{kb_0}{L_0}}$
  $h \approx 0.4$mm for $L_0 = 0.2$ and
  $h \approx 0.2$mm for $L_0 = 1$

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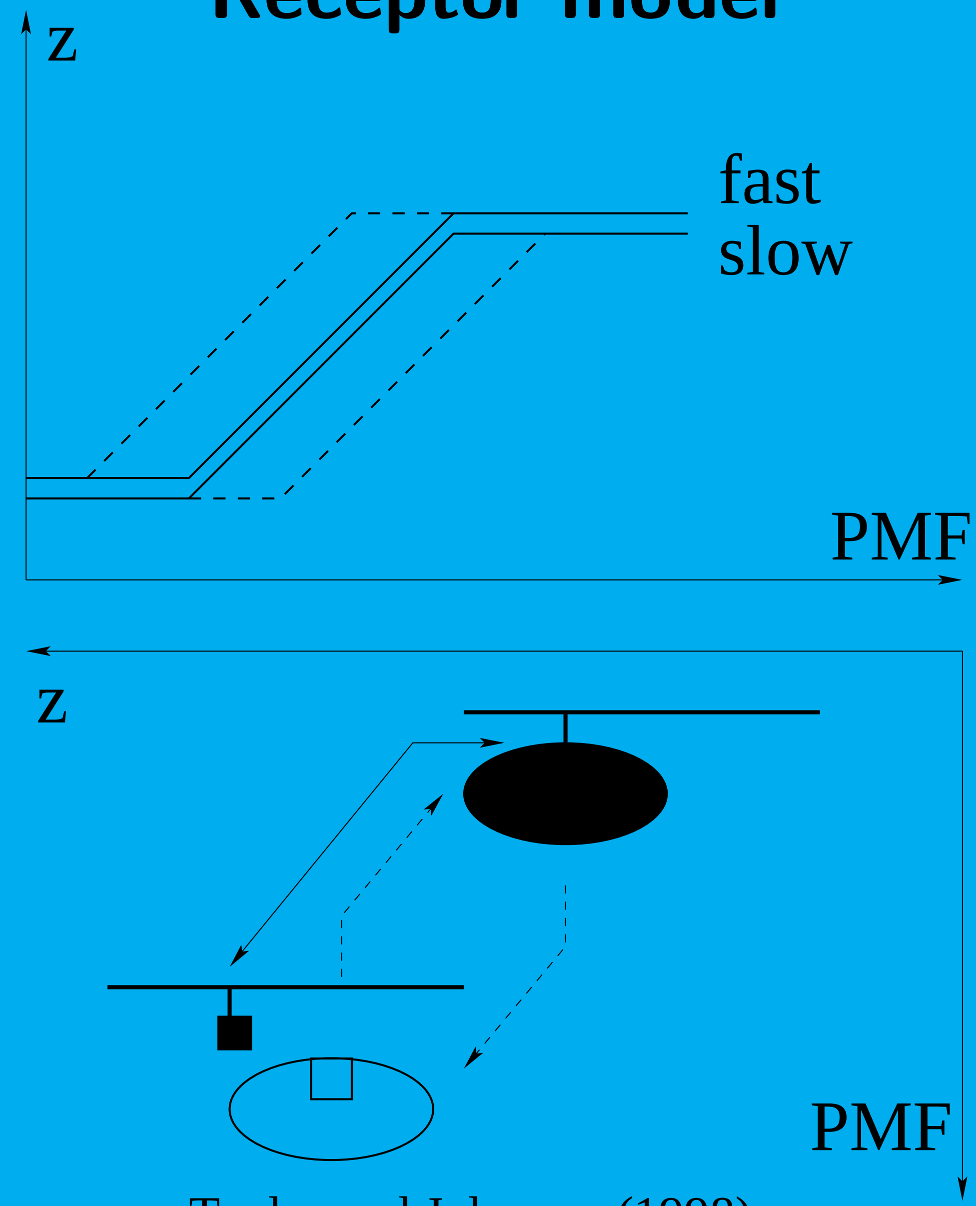
- width of bacterial band, h, is independent of L_0 in steady state and dependent on L_0 in the quasi steady state
- numerical values of d, h, bacterial density and time of pattern formation agrees with experimental values
- model allows estimates for $\tilde{L}_{max}$ and $\tilde{L}_{min}$ which are experimentally difficult to obtain

Chemotactic models

	Small spatial gradient	Large spatial gradient
Slow adaptation	Keller−Segel equation	No tractable model
Fast adapation	Advection equation	Our model

Receptor model



Taylor and Johnson (1998)

Thank you!

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Equilibrium position for fast moving part: $\tilde{z}_f = z_f^0 + c_1 p$

For slow moving part: $\tilde{z}_s = z_s^0 + c_1 p = z_f^0 - \Delta z + c_1 p$

Proton motive force: $p = \pm kt + c_0$

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Solution:

$$z_f = z_f^0 + c_0 c_1 \pm c_1 kt$$
$$z_s = z_f^0 + c_1 c_0 - \Delta z \pm c_1 k(t - \tau) + z_0 e^{-t/\tau}$$

z_f^0: ground state (fast) z_s^0: ground state (slow) τ: adaptation time