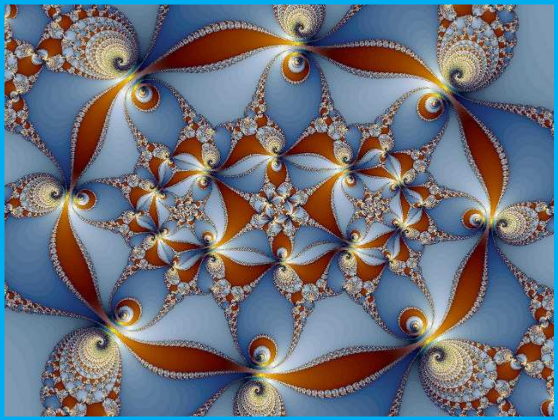
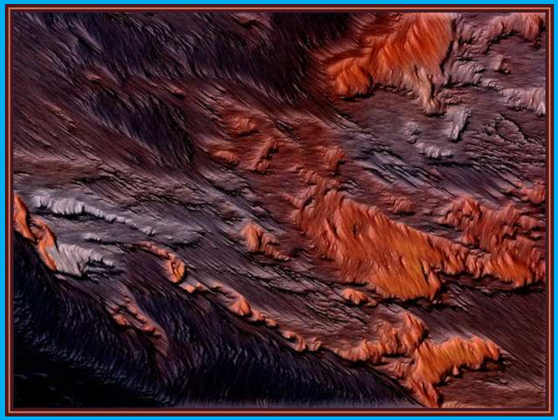
Fun with Fractals

Dr. Bori Mazzag
Redwood Empire Mathematics Tournament

March 25, 2006

Images of some fractals





What are fractals, anyway?

Important aspects of fractals:

- Self-similarity

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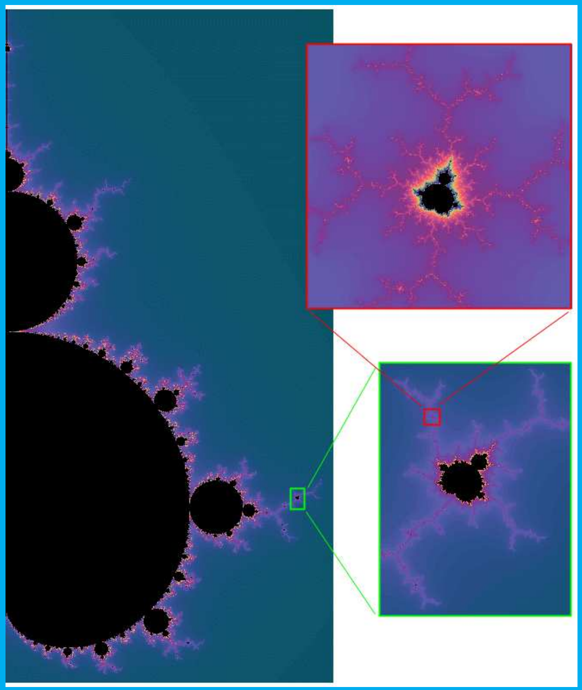
Important aspects of fractals:

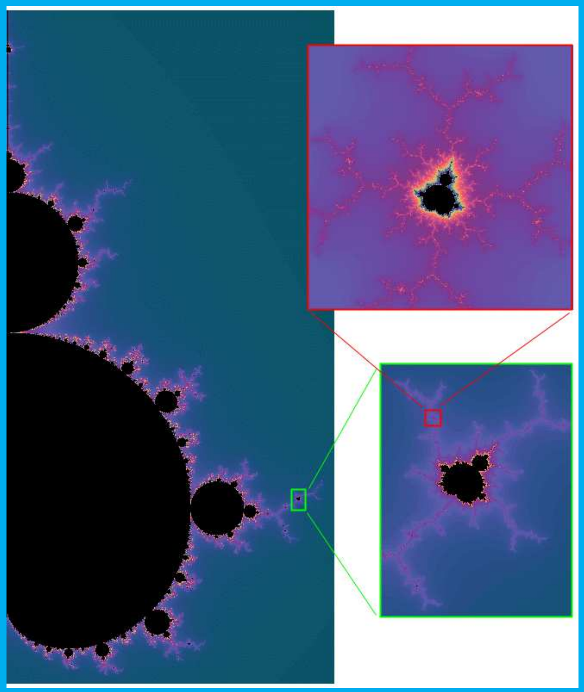
- Self-similarity
- Recursive rule to create figures
- Fractional dimensions

Self-similarity

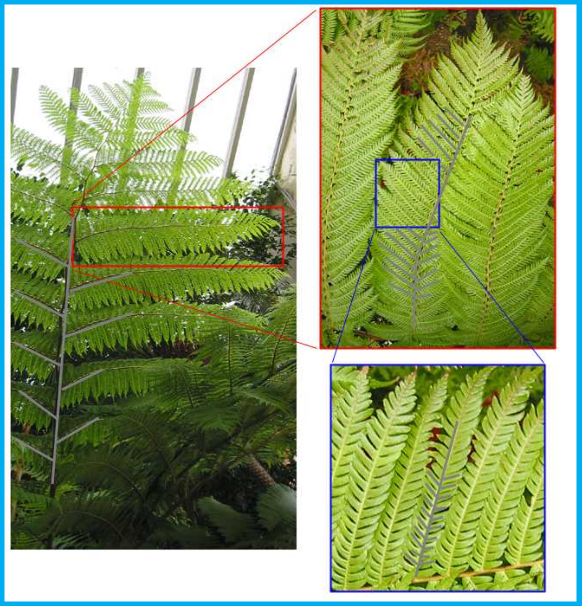


In fractals ...

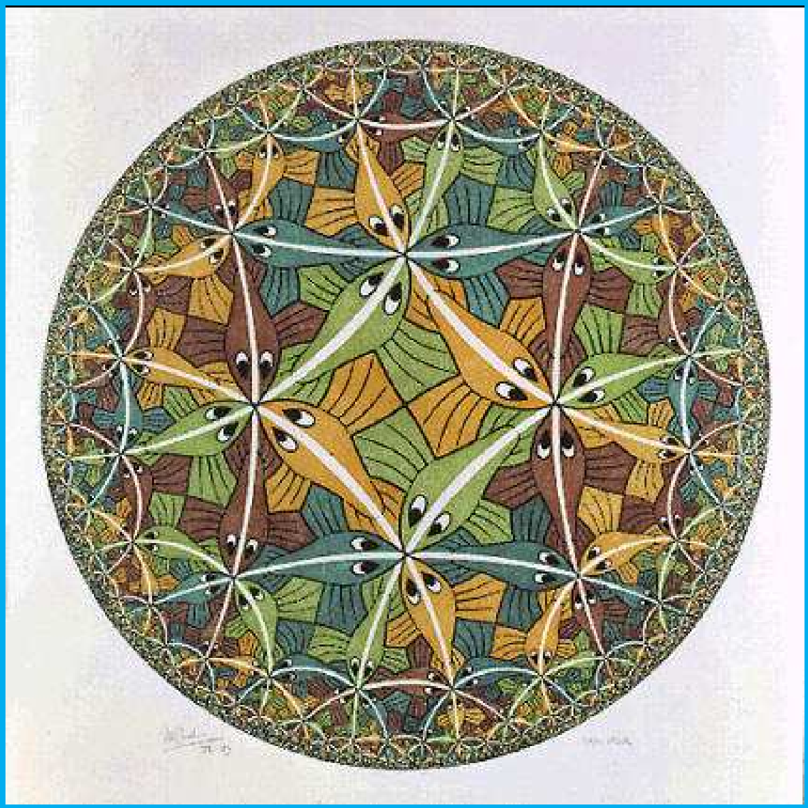
Self-similarity



In fractals ...



in nature ...



and in art.
(M.C. Escher)

Recursive rules and the Mandelbrot sequence

General recursive rule: $s_{N+1} = f(s_N)$

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Let's try it on a few starting points:

$$s_0 = 1 \rightarrow s_1 = s_0^2 + s = 1^2 + 1 = 2 \rightarrow s_2 = s_1^2 + s = 2^2 + 1 = 5$$

(**escaping sequence**)

Recursive rules and the Mandelbrot sequence

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(escaping sequence)

$$s_0 = -1 \to s_1 = (-1)^2 - 1 = 0 \to s_2 = 0^2 - 1 = -1$$

(periodic sequence)

$$s_0 = -0.75 \rightarrow s_1 = (-0.75)^2 - 0.75 = -0.19 \rightarrow$$
$$s_2 = (-0.19)^2 - 0.75 = -0.72$$

as N gets large, $s_N \rightarrow -0.5$, so -0.5 is an **attractor**

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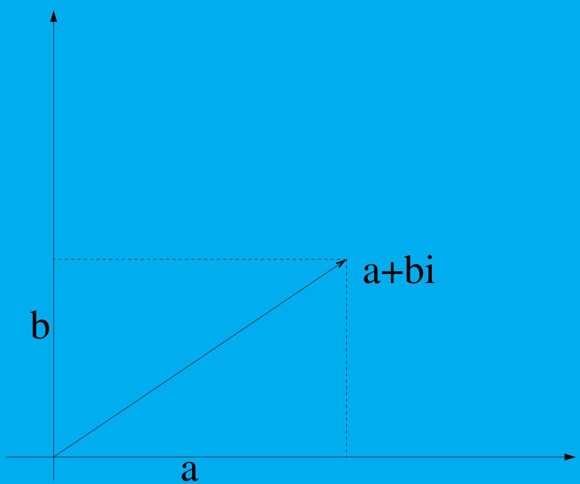

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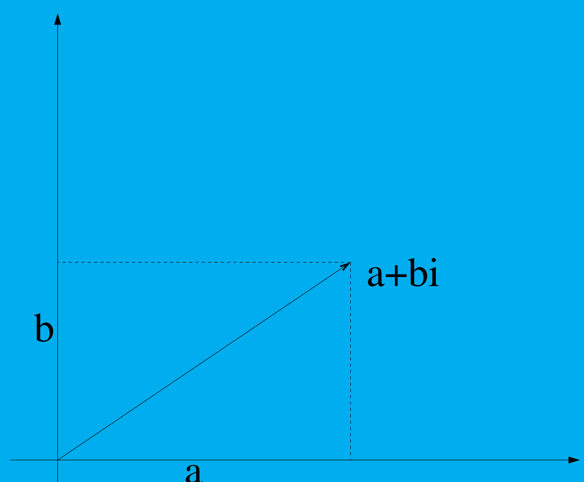
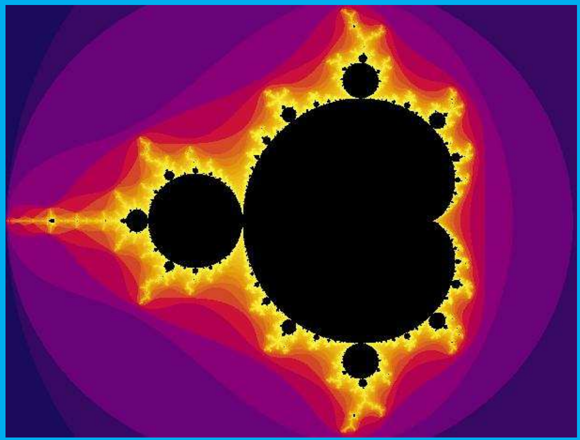
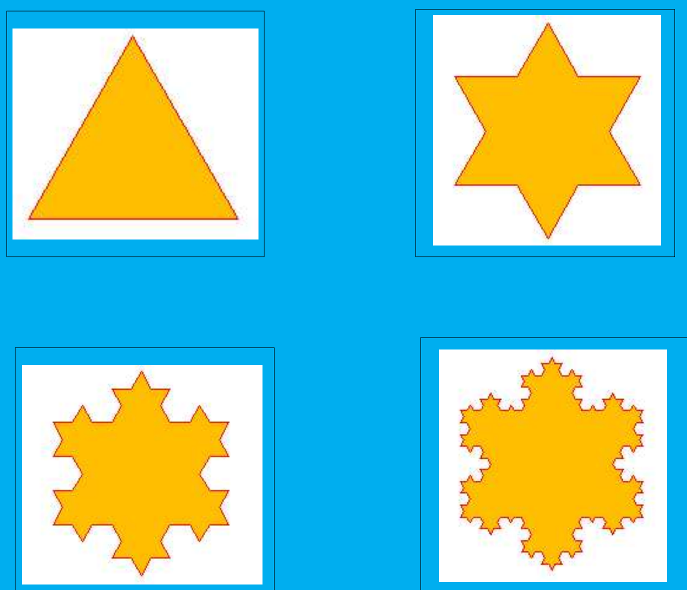




(i) seeds of periodic sequences and sequences that approach attractors are colored **black**

(ii) seeds if escaping sequences are colored

Creating a fractal – Recursion with pictures



Generating the Koch snowflake

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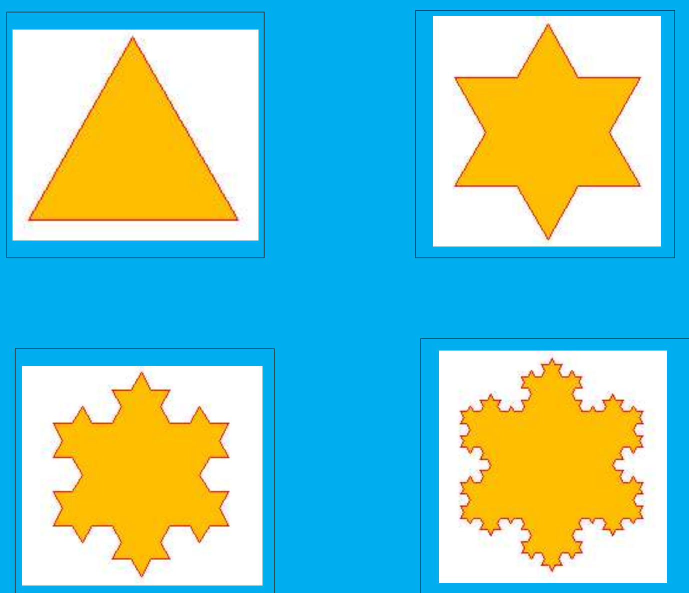

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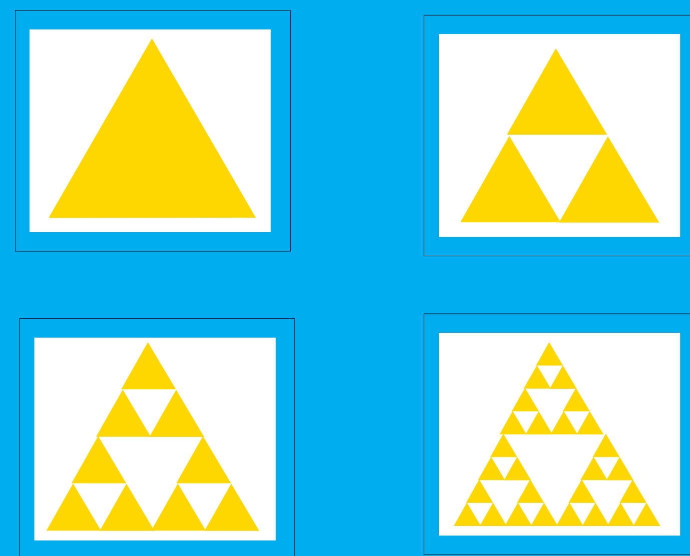
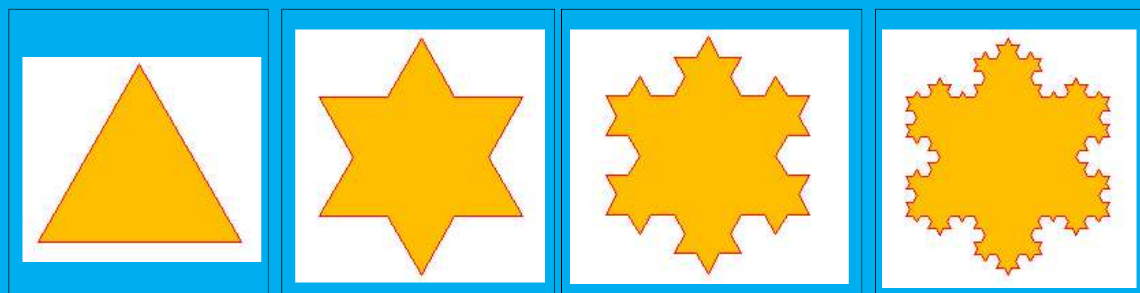


Generating Sierpinski's gasket

Calculating the perimenter of the the Koch snowflake



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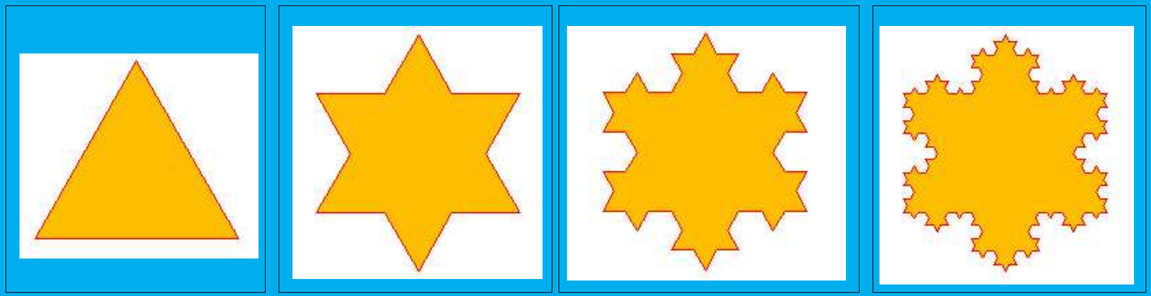

Start	Step 1	Step 2	Step 3
3	$\left(\frac{4}{3}\right) \cdot 3$	$\left(\frac{4}{3}\right)^2 \cdot 3$	$\left(\frac{4}{3}\right)^3 \cdot 3$

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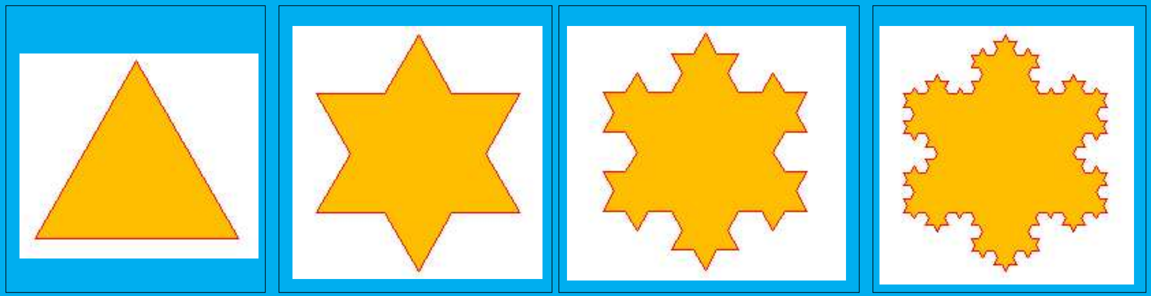

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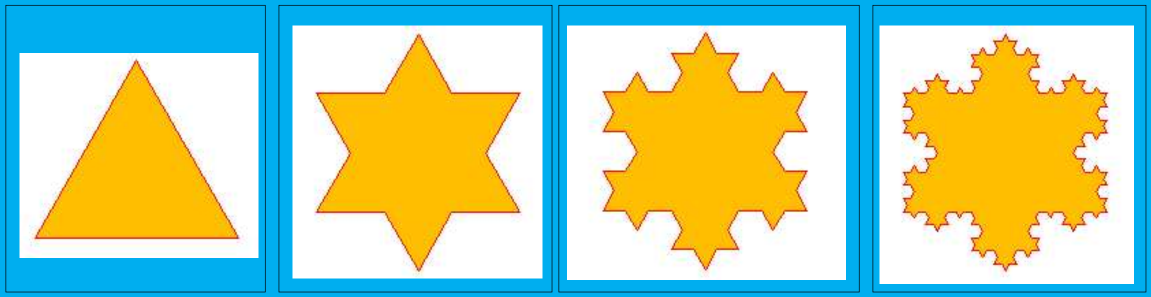
As the number of steps, N increases, so does $\left(\frac{4}{3}\right)^N$ and the perimenter becomes infinite.

The area of the Koch snowflake - activity



	Start	Step 1	Step 2	Step 3	Step N
Number of edges	3	$3 \cdot 4$			
# new triangles	-	3			
area of each new triangle	-	$\left(\frac{1}{9}\right) \cdot A$			
total new area	-	$3 \cdot \frac{1}{9} \cdot A$			
total area	A	$A + \frac{1}{3}A$			

The area of the Koch snowflake

	Start	Step 1	Step 2	Step N
Number of edges	3	$3 \cdot 4$	48	$3 \cdot 4^N$
# new triangles	-	3	12	$3 \cdot 4^{N-1}$
A of each new triangle	-	$\left(\frac{1}{9}\right) A$	$\left(\frac{1}{9}\right)^2 A$	$\left(\frac{1}{9}\right)^N A$
total new A	-	$3 \cdot \frac{1}{9} A$	$\left(\frac{4}{9}\right) \cdot \left(\frac{1}{3}\right) A$	$\left(\frac{4}{9}\right)^{N-1} \cdot \left(\frac{1}{3}\right) A$

Total new area:

$$A + \left(\frac{1}{3}\right) A + \left(\frac{4}{9}\right)\left(\frac{1}{3}\right) A + \left(\frac{4}{9}\right)^2 \left(\frac{1}{3}\right) A + \cdots + \left(\frac{4}{9}\right)^{N-1} \left(\frac{1}{3}\right) A$$

Using: $a + ar + ar^2 + \cdots + ar^{N-1} = \frac{a(r^N - 1)}{r-1}$ we get:

Area of the Koch snowflake is $\frac{8}{5}A = 1.6A$.

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The Koch snowflake has finite area and infinite perimeter!

Computing dimensions

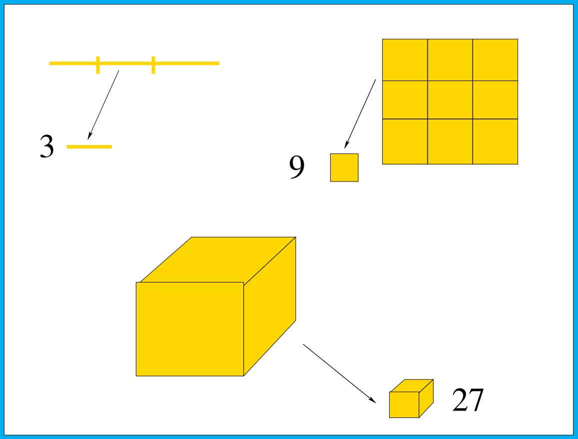
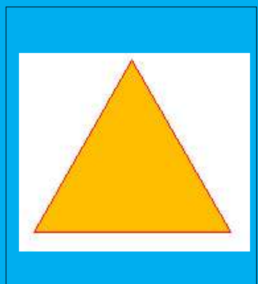
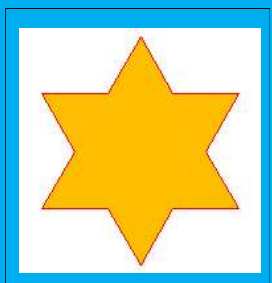
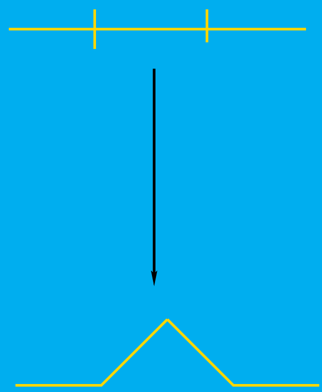


Figure	Dimension	# of copies
Line	1	3
Square	2	9
Cube	3	27
Shape	d	$n = 3^d$

Dimension of the Koch snowflake







We get 4 new line segments or 4 "self-similar copies". So, in our formula

$$4 = 3^d.$$

Compute d, the dimension:

$$d = \frac{\ln 4}{\ln 3} = 1.26$$

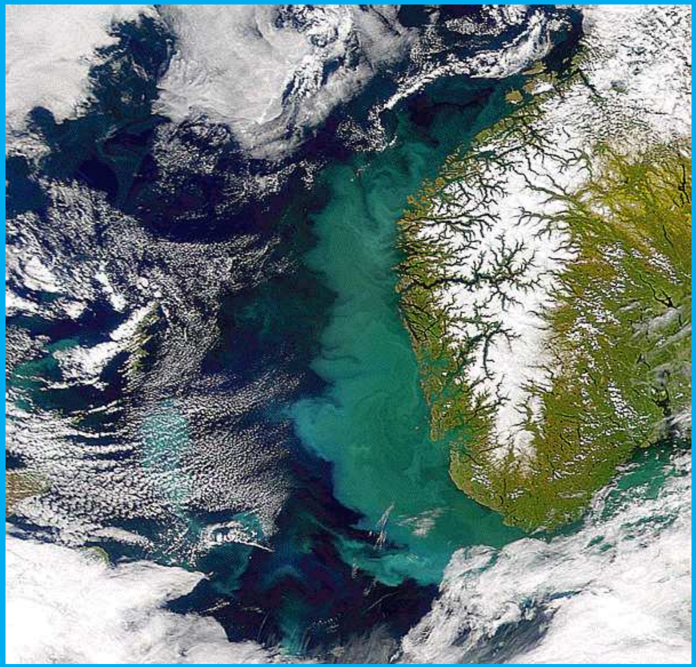
The Koch snowflake has fractional dimensions!

Applications of fractals

Ben-Jacob: bacterial growth
in stressed environments

Biological applications
(i) Bacterial growth under stress
(ii) Classifying home-range
searches
(iii) "Branching systems":
structure of blood vessels

Applications of fractals II.



Aerial view
of rivers in Norway

Mathematical and physical
science connections

(i) Connections with other areas of mathematics (dynamical systems and chaos)

(ii) material science

(iii) modeling terrains

(iv) modeling watersheds (rivers and creeks)

(v) measuring coastlines

Some references

Stunning fractal images:
http://www.fractalus.com/sylvie/fracdime.htm

Fractal-like images in nature:
http://classes.yale.edu/Fractals/Panorama/Nature/NatFracGallery/

Mathematical details on dimensions, self-similarity and connections to chaos on Bob Devaney's page Chaos in the Classroom":
http://math.bu.edu/DYSYS/chaos-game/chaos-game.html

More mathematics:
http://library.thinkquest.org/26242/full/tutorial/ch1.html

Thank you!

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