Problem Statement

Suppose small aircraft arrive at a certain airport according to Poisson process with rate \( \alpha = 8 \) per hour, so that the number of arrivals during a time period of \( t \) hours is a Poisson rv with parameter \( \lambda = 8t \)

(a) What is the probability that exactly 5 small aircraft arrive during a 1-hour period? At least 5 aircraft? At least 10 aircraft?

(b) What are the expected value and standard deviation off the number of small aircraft that arrive during a 90-min period?

(c) What is the probability that at least 20 small aircraft arrive during a 2 ½ hour period? That at most 10 arrive during this period?

Definitions, Equations and Helpful Hints

Poisson Probability Distribution is defined by the random variable \( X(t) = A \) number of successes in a continuum (time or space). The rv \( X(t) \) is said to be a Poisson distribution or \( X(t) \sim \text{Poisson}[\lambda = \alpha t] \) (pages 128-131 in text).

The parameter \( \lambda \) is defined: \( \lambda = \alpha t \)

The pmf of \( X(t) \) is defined:
\[
p[x; \lambda] = \begin{cases} 
  \frac{e^{-\lambda} \lambda^x}{x!} & x = 0, 1, 2, \ldots \\
  0 & \text{o/w}
\end{cases}
\]

The cdf of \( X(t) \) is defined:
\[
F[x; \lambda] = \sum_{x=0}^{\infty} \frac{e^{-\lambda} \lambda^x}{x!}
\]

The expected value of \( X(t) \) is defined: \( E[X(t)] = \lambda = \alpha t \)

The variance of \( X(t) \) is defined: \( V[X] = \lambda = \alpha t \)

The standard deviation is defined: \( \sigma[X(t)] = \sqrt{V[X(t)]} = \sqrt{\lambda} = \sqrt{\alpha t} \)

Notes:

• If we have any binomial experiment in which \( n \) is large and \( p \) is small \( b[x;n,p] \approx p[x;\lambda] \) where \( \lambda = np \). This approximation can be safely applied if \( n \geq 100 \), \( p \leq 0.01 \), and \( np \leq 20 \) (page 129 in text).

• Values for the cumulative distribution, \( F[X(t)] \), of the Poisson can be found in Table 2A on page 703 in the text.
Solutions

Define: \( X = \text{Number of aircraft that arrive in } t \text{ hours} \)

Given: \( \alpha = 8 \text{ aircraft/ hour} \)
\( X \sim \text{poisson}[\lambda = \alpha t = 8t] \)

(a) What is the probability that exactly 5 small aircraft arrive during a 1-hour period? At least 5 aircraft? At least 10 aircraft? (*means see note)

- This problem has three parts. Each part involves using the Poisson distribution pmf definition however, \( x \) varies in each part. The first part of the question is solved using the Poisson distribution pmf definition, where \( t = 1 \text{ hour and } x = 5 \text{ aircraft.} \)

\[ \lambda = \alpha t = \left(8 \frac{\text{aircraft}}{\text{hour}}\right)(1 \text{ hour}) \]

\[ P[X (t = 1) = 5] = \frac{e^{-\lambda} \lambda^5}{5!} = \frac{e^{-8}(8)^5}{5!} = 0.092 \]

- Next, the problem remains the same except \( x \geq 5 \). The complement is used to find the solution:

\[ \lambda = \alpha t = \left(8 \frac{\text{aircraft}}{\text{hour}}\right)(1 \text{ hour}) \]

\[ P[X (t = 1) \geq 5] = 1 - F(4) = 1 - \sum_{x=0}^{4} \frac{e^{-\lambda} \lambda^x}{x!} \]

\[ = 1 - \sum_{x=0}^{4} \frac{e^{-8}8^x}{x!} = 1 - 0.1* = 0.9 \]

* From Table 2A page 703 in text.

- Again the problem is similar except \( x \geq 10 \). The complement is also used to find the solution:

\[ \lambda = \alpha t = \left(8 \frac{\text{aircraft}}{\text{hour}}\right)(1 \text{ hour}) \]

\[ P[X (t = 1) \geq 10] = 1 - F(9) = 1 - \sum_{x=0}^{9} \frac{e^{-\lambda} \lambda^x}{x!} \]

\[ = 1 - \sum_{x=0}^{9} \frac{e^{-8}8^x}{x!} = 1 - 0.717* = 0.283 \]

* From Table 2A page 703 in text.
(b) What are the expected value and standard deviation of the number of small aircraft that arrive during a 90-min period?

- Note that 90min = 1.5 hours. The expected value of the Poisson distribution is calculated from the definition mentioned in Definitions, Equations and Helpful Hints section:

\[ E[X \mid t = 1.50] = \lambda = \alpha t = \left( \frac{8 \text{ aircraft \ hour}}{\text{hour}} \right) (1.5 \text{ hours}) = 12 \text{ aircraft} \]

- The standard deviation of this Poisson distribution is calculated by:

\[ \sigma[X] = \sqrt{\lambda} = \sqrt{12 \text{ aircraft}^2} = 3.4641 \text{ aircraft} \]

(c) What is the probability that at least 20 small aircraft arrive during a 2 ½ hour period? That at most 10 arrive during this period?

- This problem has two parts. Each part utilizes the Poisson distribution pmf definition however the value of \( x \) varies. In the first part \( t = 2.5 \) hours and \( x \geq 20 \). The complement is used to find the solution.

\[ \lambda = \alpha t = \left( \frac{8 \text{ aircraft \ hour}}{\text{hour}} \right) (2.5 \text{ hours}) = 20 \text{ aircraft} \]

\[ P[X \mid t = 2.5] \geq 20] = 1 - F(19) \]

\[ = 1 - \sum_{x=6}^{19} \frac{e^{-\lambda} \lambda^x}{x!} = 1 - \sum_{x=0}^{19} \frac{e^{-20} 20^x}{x!} \]

\[ = 1 - 0.47* = 0.53 \]

* From Table 2A page 703 in text.

- Now \( x \leq 10 \) and Poisson distribution pmf definition is used to find the solution.

\[ P[X \mid t = 2.5] \leq 10] = F(10) \]

\[ = \sum_{x=0}^{10} \frac{e^{-\lambda} \lambda^x}{x!} = \sum_{x=0}^{10} \frac{e^{-20} 20^x}{x!} = 0.011* \]

* From Table 2A page 703 in text.
Using Microsoft EXCEL

Microsoft’s EXCEL spreadsheet contains intrinsic functions to solve Binomial and Poisson distribution pmf problems. To use this property of EXCEL, click on to a cell in the spreadsheet. This cell will contain the result. The intrinsic functions are contained in the function wizard menu. This menu can be accessed by clicking on the \( f(x) \) button on the toolbar or by pulling down the “Insert” menu and scrolling down to “Function”. A function window should appear.

For Binomial distribution applications

- Select “Statistical” in the function category and “BINODIST” in the function name category. Click “OK”.
- A window should appear asking for the following information:
  - \( \text{Number}_s \) = the number of successful trials, \( x \)
  - \( \text{Trials} \) = the number of trials, \( n \)
  - \( \text{Probability}_s \) = the probability of a success, \( p(x) \)
  - \( \text{Cumulative} \) = true or false. True if the cumulative distribution function is to be calculated. False for the converse.

For Poisson distribution applications

- Select “Statistical” in the function category and “POISSON” in the function name category. Click “OK”.
- A window should appear asking for the following information:
  - \( \text{X} \) = the number of events, \( n \)
  - \( \text{Mean} \) = the expected value of, \( X \)
  - \( \text{Cumulative} \) = true or false. True if the cumulative distribution function is to be calculated. False for the converse.

Following these steps and correctly answering the function requirements should give a result. For example, Figure 1 is a graph comparing different Poisson distribution pmfs with varying values of \( \lambda \). The pmf values in the following graph are the results from the EXCEL intrinsic function “POISSON”. The graph’s trend shows: as \( \lambda \) increases the maximal probability occurs with an increased number of successes in a time period. The pmfs also become increasingly more distributed as \( \lambda \) increases.
Figure 1. Comparison of the Poisson distribution pmf with varying values of $\lambda$. 

The graph shows the probability mass function (PMF) for different values of $\lambda$ and $t$. The x-axis represents the number of aircraft that arrive in $t$ hours, and the y-axis shows the probability $P[X(t)]$. The figure illustrates how the shape of the distribution changes with different values of $\lambda$. 

**Legend**

- **$\lambda = 8$ (t=1)**: Diamond markers
- **$\lambda = 12$ (t=1.5)**: Square markers
- **$\lambda = 20$ (t=2.5)**: Triangle markers